Chapter 6. Right Triangle Trigonometry

Chapter Outline

6.1 The Pythagorean Theorem
6.2 Converse of the Pythagorean Theorem
6.3 Using Similar Right Triangles
6.4 Tangent, Sine and Cosine
6.5 Inverse Trigonometric Ratios
6.6 Right Triangle Trigonometry Review

Chapter 8 explores right triangles in far more depth than Chapters 4 and 5. Recall that a right triangle is a triangle with exactly one right angle. In this chapter, we will first prove the Pythagorean Theorem and its converse, followed by analyzing the sides of certain types of triangles. Then, we will introduce trigonometry, which starts with the tangent, sine and cosine ratios. Finally, we will extend sine and cosine to any triangle, through the Law of Sines and the Law of Cosines.
6.1 The Pythagorean Theorem

Learning Objectives

- Prove and use the Pythagorean Theorem.
- Identify common Pythagorean triples.
- Use the Pythagorean Theorem to find the area of isosceles triangles.
- Use the Pythagorean Theorem to derive the distance formula on a coordinate grid.

Review Queue

1. Draw a right scalene triangle.
2. Draw an isosceles right triangle.
3. Simplify the radical.
   a. \( \sqrt{50} \)
   b. \( \sqrt{27} \)
   c. \( \sqrt{272} \)
4. Perform the indicated operations on the following numbers. Simplify all radicals.
   a. \( 2 \sqrt{10} + \sqrt{160} \)
   b. \( 5 \sqrt{6} \cdot 4 \sqrt{18} \)
   c. \( \sqrt{8} \cdot 12 \sqrt{2} \)

Know What? All televisions dimensions refer to the diagonal of the rectangular viewing area. Therefore, for a 52” TV, 52” is the length of the diagonal. High Definition Televisions (HDTVs) have sides in the ratio of 16:9. What is the length and width of a 52” HDTV? What is the length and width of an HDTV with a \( y’’ \) long diagonal?

The Pythagorean Theorem

We have used the Pythagorean Theorem already in this text, but we have never proved it. Recall that the sides of a right triangle are called legs (the sides of the right angle) and the side opposite the right angle is the hypotenuse. For the Pythagorean Theorem, the legs are “\( a \)” and “\( b \)” and the hypotenuse is “\( c \)”. 
Pythagorean Theorem: Given a right triangle with legs of lengths $a$ and $b$ and a hypotenuse of length $c$, then $a^2 + b^2 = c^2$.

There are several proofs of the Pythagorean Theorem. We will provide one proof within the text and two others in the review exercises.

**Investigation 8-1: Proof of the Pythagorean Theorem**

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in square and a right triangle with legs of 3 and 4 inches.
2. Cut out the triangle and square and arrange them like the picture on the right.

3. This theorem relies on area. Recall from a previous math class, that the area of a square is length times width. But, because the sides are the same you can rewrite this formula as $A_{\text{square}} = \text{length} \times \text{width} = \text{side} \times \text{side} = \text{side}^2$. So, the Pythagorean Theorem can be interpreted as $(\text{square with side } a)^2 + (\text{square with side } b)^2 = (\text{square with side } c)^2$. In this Investigation, the sides are 3, 4 and 5 inches. What is the area of each square?
4. Now, we know that $9 + 16 = 25$, or $3^2 + 4^2 = 5^2$. Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

**Another Proof of the Pythagorean Theorem**

This proof is “more formal,” meaning that we will use letters, $a, b, c$ to represent the sides of the right triangle. In this particular proof, we will take four right triangles, with legs $a$ and $b$ and hypotenuse $c$ and make the areas equal.
6.1. The Pythagorean Theorem

For two animated proofs, go to http://www.mathsisfun.com/pythagoras.html and scroll down to “And You Can Prove the Theorem Yourself.”

Using the Pythagorean Theorem

The Pythagorean Theorem can be used to find a missing side of any right triangle, to prove that three given lengths can form a right triangle, to find Pythagorean Triples, to derive the Distance Formula, and to find the area of an isosceles triangle. Here are several examples. Simplify all radicals.

Example 1: Do 6, 7, and 8 make the sides of a right triangle?

Solution: Plug in the three numbers into the Pythagorean Theorem. The largest length will always be the hypotenuse. $6^2 + 7^2 = 36 + 49 = 85 \neq 8^2$. Therefore, these lengths do not make up the sides of a right triangle.

Example 2: Find the length of the hypotenuse of the triangle below.

Solution: Let’s use the Pythagorean Theorem. Set $a$ and $b$ equal to 8 and 15 and solve for $c$, the hypotenuse.

\[
8^2 + 15^2 = c^2 \\
64 + 225 = c^2 \\
289 = c^2 \quad \text{Take the square root of both sides.} \\
17 = c
\]
When you take the square root of an equation, usually the answer is +17 or -17. Because we are looking for length, we only use the positive answer. **Length is never negative.**

**Example 3:** Find the missing side of the right triangle below.

![Right Triangle Diagram]

**Solution:** Here, we are given the hypotenuse and a leg. Let’s solve for $b$.

\[ 7^2 + b^2 = 14^2 \]
\[ 49 + b^2 = 196 \]
\[ b^2 = 147 \]
\[ b = \sqrt{147} = \sqrt{7 \cdot 7 \cdot 3} = 7 \sqrt{3} \]

**Example 4:** What is the diagonal of a rectangle with sides 10 and $16 \sqrt{5}$?

![Rectangle Diagram]

**Solution:** For any square and rectangle, you can use the Pythagorean Theorem to find the length of a diagonal. Plug in the sides to find $d$.

\[ 10^2 + \left(16 \sqrt{5}\right)^2 = d^2 \]
\[ 100 + 1280 = d^2 \]
\[ 1380 = d^2 \]
\[ d = \sqrt{1380} = 2 \sqrt{345} \]

**Pythagorean Triples**

In Example 2, the sides of the triangle were 8, 15, and 17. This combination of numbers is referred to as a **Pythagorean triple.**

**Pythagorean Triple:** A set of three whole numbers that makes the Pythagorean Theorem true.
The most frequently used Pythagorean triple is 3, 4, 5, as in Investigation 8-1. Any multiple of a Pythagorean triple is also considered a triple because it would still be three whole numbers. Therefore, 6, 8, 10 and 9, 12, 15 are also sides of a right triangle. Other Pythagorean triples are:

3, 4, 5     5, 12, 13     7, 24, 25     8, 15, 17

There are infinitely many Pythagorean triples. To see if a set of numbers makes a triple, plug them into the Pythagorean Theorem.

**Example 5:** Is 20, 21, 29 a Pythagorean triple?

**Solution:** If \( 20^2 + 21^2 \) is equal to \( 29^2 \), then the set is a triple.

\[
20^2 + 21^2 = 400 + 441 = 841 \\
29^2 = 841
\]

Therefore, 20, 21, and 29 is a Pythagorean triple.

---

**Area of an Isosceles Triangle**

There are many different applications of the Pythagorean Theorem. One way to use the Pythagorean Theorem is to identify the heights in isosceles triangles so you can calculate the area. The area of a triangle is \( \frac{1}{2} bh \), where \( b \) is the base and \( h \) is the height (or altitude).

If you are given the base and the sides of an isosceles triangle, you can use the Pythagorean Theorem to calculate the height.

**Example 6:** What is the area of the isosceles triangle?

**Solution:** First, draw the altitude from the vertex between the congruent sides, which will bisect the base (Isosceles Triangle Theorem). Then, find the length of the altitude using the Pythagorean Theorem.
\[ 7^2 + h^2 = 9^2 \]
\[ 49 + h^2 = 81 \]
\[ h^2 = 32 \]
\[ h = \sqrt{32} = 4\sqrt{2} \]

Now, use \( h \) and \( b \) in the formula for the area of a triangle.

\[
A = \frac{1}{2}bh = \frac{1}{2}(14)(4\sqrt{2}) = 28\sqrt{2} \text{ units}^2
\]

**The Distance Formula**

Another application of the Pythagorean Theorem is the Distance Formula. We have already been using the Distance Formula in this text, but we can prove it here.

First, draw the vertical and horizontal lengths to make a right triangle. Then, use the differences to find these distances.

Now that we have a right triangle, we can use the Pythagorean Theorem to find \( d \).
6.1. The Pythagorean Theorem

Distance Formula: The distance \( A(x_1, y_1) \) and \( B(x_2, y_2) \) is \[ d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}. \]

Example 7: Find the distance between (1, 5) and (5, 2).

Solution: Make \( A(1,5) \) and \( B(5,2) \). Plug into the distance formula.

\[ d = \sqrt{(1-5)^2 + (5-2)^2} \]
\[ = \sqrt{(-4)^2 + (3)^2} \]
\[ = \sqrt{16+9} = \sqrt{25} = 5 \]

You might recall that the distance formula was presented as \[ d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}, \] with the first and second points switched. It does not matter which point is first as long as \( x \) and \( y \) are both first in each parenthesis. In Example 7, we could have switched \( A \) and \( B \) and would still get the same answer.

\[ d = \sqrt{(5-1)^2 + (2-5)^2} \]
\[ = \sqrt{(4)^2 + (-3)^2} \]
\[ = \sqrt{16+9} = \sqrt{25} = 5 \]

Also, just like the lengths of the sides of a triangle, distances are always positive.

Know What? Revisited To find the length and width of a 52” HDTV, plug in the ratios and 52 into the Pythagorean Theorem. We know that the sides are going to be a multiple of 16 and 9, which we will call \( n \).

\[ (16n)^2 + (9n)^2 = 52^2 \]
\[ 256n^2 + 81n^2 = 2704 \]
\[ 337n^2 = 2704 \]
\[ n^2 = 8.024 \]
\[ n = 2.83 \]
Therefore, the dimensions of the TV are $16(2.83''\text{})$ by $9(2.833''\text{})$, or $45.3''$ by $25.5''$. If the diagonal is $y''$ long, it would be $n \sqrt{337}''$ long. The extended ratio is $9 : 16 : \sqrt{337}$.

**Review Questions**

Find the length of the missing side. Simplify all radicals.

7. If the legs of a right triangle are 10 and 24, then the hypotenuse is ____________.
8. If the sides of a rectangle are 12 and 15, then the diagonal is ____________.
9. If the legs of a right triangle are $x$ and $y$, then the hypotenuse is ____________.
10. If the sides of a square are 9, then the diagonal is ____________.
6.1. The Pythagorean Theorem

Determine if the following sets of numbers are Pythagorean Triples.

11. 12, 35, 37
12. 9, 17, 18
13. 10, 15, 21
14. 11, 60, 61
15. 15, 20, 25
16. 18, 73, 75

Find the area of each triangle below. Simplify all radicals.

20. (-1, 6) and (7, 2)
21. (10, -3) and (-12, -6)
22. (1, 3) and (-8, 16)

Find the length between each pair of points.

23. What are the length and width of a 42” HDTV? Round your answer to the nearest tenth.
24. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a 42” Standard definition TV? Round your answer to the nearest tenth.

25. Challenge An equilateral triangle is an isosceles triangle. If all the sides of an equilateral triangle are $s$, find the area, using the technique learned in this section. Leave your answer in simplest radical form.

26. Find the area of an equilateral triangle with sides of length 8.

Pythagorean Theorem Proofs
The first proof below is similar to the one done earlier in this lesson. Use the picture below to answer the following questions.

27. Find the area of the square with sides \((a + b)\).
28. Find the sum of the areas of the square with sides \(c\) and the right triangles with legs \(a\) and \(b\).
29. The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

Major General James A. Garfield (and former President of the U.S) is credited with deriving this next proof of the Pythagorean Theorem using a trapezoid.

30. Find the area of the trapezoid using the trapezoid area formula: \(A = \frac{1}{2}(b_1 + b_2)h\)
31. Find the sum of the areas of the three right triangles in the diagram.
32. The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

**Review Queue Answers**

1. [Image of a right triangle with a small square at the bottom right corner]
3. Answers:
   a. $\sqrt{50} = \sqrt{25 \cdot 2} = 5 \sqrt{2}$
   b. $\sqrt{27} = \sqrt{9 \cdot 3} = 3 \sqrt{3}$
   c. $\sqrt{272} = \sqrt{16 \cdot 17} = 4 \sqrt{17}$

4. Answers:
   a. $2 \sqrt{10} + \sqrt{160} = 2 \sqrt{10} + 4 \sqrt{10} = 4 \sqrt{10}$
   b. $5 \sqrt{6} \cdot 4 \sqrt{18} = 5 \sqrt{6} \cdot 12 \sqrt{2} = 60 \sqrt{12} = 120 \sqrt{3}$
   c. $\sqrt{8} \cdot 12 \sqrt{2} = 12 \sqrt{16} = 12 \cdot 4 = 48$
6.2 Converse of the Pythagorean Theorem

Learning Objectives

- Understand the converse of the Pythagorean Theorem.
- Identify acute and obtuse triangles from side measures.

Review Queue

1. Determine if the following sets of numbers are Pythagorean triples.
   a. 14, 48, 50
   b. 9, 40, 41
   c. 12, 43, 44

2. Do the following lengths make a right triangle? How do you know?
   a. $\sqrt{5}, 3, \sqrt{14}$
   b. 6, $2\sqrt{3}, 8$
   c. $3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Know What? A friend of yours is designing a building and wants it to be rectangular. One wall 65 ft. long and the other is 72 ft. long. How can he ensure the walls are going to be perpendicular?

Converse of the Pythagorean Theorem

In the last lesson, you learned about the Pythagorean Theorem and how it can be used. The converse of the Pythagorean Theorem is also true. We touched on this in the last section with Example 1.

**Pythagorean Theorem Converse:** If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any of the triangle’s angle measurements.
6.2. Converse of the Pythagorean Theorem

**Example 1:** Determine if the triangles below are right triangles.

a)

![Diagram of a triangle with sides 8, 16, and 8√5]

**Solution:** Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest sides represent \( c \), in the equation.

\[
a^2 + b^2 = c^2
\]

\[
8^2 + 16^2 = (8\sqrt{5})^2
\]

\[
64 + 256 = 64 \cdot 5
\]

\[
320 = 320
\]

The triangle is a right triangle.

b)

![Diagram of a triangle with sides 22, 24, and 26]

**Solution:** Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest sides represent \( c \), in the equation.

\[
a^2 + b^2 = c^2
\]

\[
22^2 + 24^2 \neq 26^2
\]

\[
484 + 576 \neq 676
\]

\[
1060 \neq 676
\]

The triangle is not a right triangle.

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**Identifying Acute and Obtuse Triangles**

We can extend the converse of the Pythagorean Theorem to determine if a triangle has an obtuse angle or is acute. We know that if the sum of the squares of the two smaller sides equals the square of the larger side, then the triangle is right. We can also interpret the outcome if the sum of the squares of the smaller sides does not equal the square of the third.

**Theorem 8-3:** If the sum of the squares of the two shorter sides in a right triangle is greater than the square of the longest side, then the triangle is acute.

**Theorem 8-4:** If the sum of the squares of the two shorter sides in a right triangle is less than the square of the longest side, then the triangle is obtuse.

In other words: The sides of a triangle are \( a, b, \) and \( c \) and \( c > b \) and \( c > a \).

If \( a^2 + b^2 > c^2 \), then the triangle is acute.

If \( a^2 + b^2 = c^2 \), then the triangle is right.

If \( a^2 + b^2 < c^2 \), then the triangle is obtuse.
Proof of Theorem 8-3

Given: In \( \triangle ABC, a^2 + b^2 > c^2 \), where \( c \) is the longest side.

In \( \triangle LMN, \angle N \) is a right angle.

Prove: \( \triangle ABC \) is an acute triangle. (all angles are less than 90°)

\[ \begin{array}{|c|c|}
\hline
\text{Statement} & \text{Reason} \\
\hline
1. In \( \triangle ABC, a^2 + b^2 > c^2 \), and \( c \) is the longest side. In \( \triangle LMN, \angle N \) is a right angle. & \text{Given} \\
2. \( a^2 + b^2 = h^2 \) & \text{Pythagorean Theorem} \\
3. \( c^2 < h^2 \) & \text{Transitive PoE} \\
4. \( c < h \) & \text{Take the square root of both sides} \\
5. \( \angle C \) is the largest angle in \( \triangle ABC \). & \text{The largest angle is opposite the longest side.} \\
6. \( m\angle N = 90^\circ \) & \text{Definition of a right angle} \\
7. \( m\angle C < m\angle N \) & \text{SSS Inequality Theorem} \\
8. \( m\angle C < 90^\circ \) & \text{Transitive PoE} \\
9. \( \angle C \) is an acute angle. & \text{Definition of an acute angle} \\
10. \( \triangle ABC \) is an acute triangle. & \text{If the largest angle is less than 90°, then all the angles are less than 90°}. \\
\hline
\end{array} \]

The proof of Theorem 8-4 is very similar and is in the review questions.

Example 2: Determine if the following triangles are acute, right or obtuse.

a)
6.2. Converse of the Pythagorean Theorem

Solution: Set the shorter sides in each triangle equal to \( a \) and \( b \) and the longest side equal to \( c \).

a) \( 6^2 + (3 \sqrt{5})^2 \leq 8^2 \)
\[ 36 + 45 \leq 64 \]
\[ 81 > 64 \]
The triangle is acute.

b) \( 15^2 + 14^2 \geq 21^2 \)
\[ 225 + 196 \geq 441 \]
\[ 421 < 441 \]
The triangle is obtuse.

Example 3: Graph \( A(-4, 1), B(3, 8), \) and \( C(9, 6) \). Determine if \( \triangle ABC \) is acute, obtuse, or right.

Solution: This looks like an obtuse triangle, but we need proof to draw the correct conclusion. Use the distance formula to find the length of each side.

\[
AB = \sqrt{(-4 - 3)^2 + (1 - 8)^2} = \sqrt{49 + 49} = \sqrt{98} = 7 \sqrt{2}
\]
\[
BC = \sqrt{(3 - 9)^2 + (8 - 6)^2} = \sqrt{36 + 4} = \sqrt{40} = 2 \sqrt{10}
\]
\[
AC = \sqrt{(-4 - 9)^2 + (1 - 6)^2} = \sqrt{169 + 25} = \sqrt{194}
\]

Now, let’s plug these lengths into the Pythagorean Theorem.

\[
(\sqrt{98})^2 + (\sqrt{40})^2 \leq (\sqrt{194})^2
\]
\[98 + 40 \leq 194 \]
\[138 < 194 \]
\( \triangle ABC \) is an obtuse triangle.

**Know What? Revisited** To make the walls perpendicular, find the length of the diagonal.

\[
65^2 + 72^2 = c^2 \\
4225 + 5184 = c^2 \\
9409 = c^2 \\
97 = c
\]

In order to make the building rectangular, both diagonals must be 97 feet.

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**Review Questions**

1. The two shorter sides of a triangle are 9 and 12.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?

2. The two longer sides of a triangle are 24 and 25.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?

3. The lengths of the sides of a triangle are 8\(x\), 15\(x\), and 17\(x\). Determine if the triangle is acute, right, or obtuse.

Determine if the following lengths make a right triangle.

4. 15, 20, 25
5. 20, 25, 30
6. \(8\sqrt{3}, 6, 2\sqrt{39}\)

Determine if the following triangles are acute, right or obtuse.

7. 7, 8, 9
8. 14, 48, 50
9. 5, 12, 15
10. 13, 84, 85
11. 20, 20, 24
12. 35, 40, 51
13. 39, 80, 89
14. 20, 21, 38
15. 48, 55, 76

Graph each set of points and determine if \( \triangle ABC \) is acute, right, or obtuse.

16. \( A(3, -5), B(-5, -8), C(-2, 7) \)
17. \( A(5, 3), B(2, -7), C(-1, 5) \)
18. **Writing** Explain the two different ways you can show that a triangle in the coordinate plane is a right triangle.
The figure to the right is a rectangular prism. All sides (or faces) are either squares (the front and back) or rectangles (the four around the middle). All sides are perpendicular.

19. Find $c$.
20. Find $d$.

21. **Writing** Explain why $m\angle A = 90^\circ$.
22. Fill in the blanks for the proof of Theorem 8-4.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. In $\triangle ABC$, $a^2 + b^2 &lt; c^2$, and $c$ is the longest side. In $\triangle LMN$, $\angle N$ is a right angle.</td>
<td></td>
</tr>
<tr>
<td>2. $a^2 + b^2 = h^2$</td>
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</tr>
<tr>
<td>3. $c^2 &gt; h^2$</td>
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<td>4.</td>
<td></td>
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<tr>
<td>5. $\angle C$ is the largest angle in $\triangle ABC$.</td>
<td></td>
</tr>
<tr>
<td>6. $m\angle N = 90^\circ$</td>
<td></td>
</tr>
<tr>
<td>7. $m\angle C &gt; m\angle N$</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>Transitive PoE</td>
</tr>
<tr>
<td>9. $\angle C$ is an obtuse angle.</td>
<td></td>
</tr>
<tr>
<td>304. $\triangle ABC$ is an obtuse triangle.</td>
<td></td>
</tr>
</tbody>
</table>
Given $\overline{AB}$, with $A(3, 3)$ and $B(2, -3)$ determine whether the given point $C$ in problems 23-25 makes an acute, right or obtuse triangle.

23. $C(3, -3)$
24. $C(4, -1)$
25. $C(5, -2)$

Given $\overline{AB}$, with $A(-2, 5)$ and $B(1, -3)$ find at least two possible points, $C$, such that $\triangle ABC$ is

26. right, with right $\angle C$.
27. acute, with acute $\angle C$.
28. obtuse, with obtuse $\angle C$.
29. **Construction**
   a. Draw $\overline{AB}$, such that $AB = 3$ in.
   b. Draw $\overrightarrow{AD}$ such that $\angle BAD < 90^\circ$.
   c. Construct a line through $B$ which is perpendicular to $\overrightarrow{AD}$, label the intersection $C$.
   d. $\triangle ABC$ is a right triangle with right $\angle C$.

30. Is the triangle you made unique? In other words, could you have multiple different outcomes with the same $AB$? Why or why not? You may wish to experiment to find out.
31. Why do the instructions specifically require that $\angle BAD < 90^\circ$?
32. Describe how this construction could be changed so that $\angle B$ is the right angle in the triangle.

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**Review Queue Answers**

1. Answers:
   a. Yes
   b. Yes
   c. No
2. Answers:
   a. Yes
   b. No
   c. Yes
6.3 Using Similar Right Triangles

Learning Objectives

- Identify similar triangles inscribed in a larger triangle.
- Evaluate the geometric mean.
- Find the length of an altitude or leg using the geometric mean.

Review Queue

1. If two triangles are right triangles, does that mean they are similar? Explain.
2. If two triangles are isosceles right triangles, does that mean they are similar? Explain.
3. Solve the ratio: $\frac{3}{x} = \frac{x}{27}$.
4. If the legs of an isosceles right triangle are 4, find the length of the hypotenuse. Draw a picture and simplify the radical.

Know What? In California, the average home price increased 21.3% in 2004 and another 16.0% in 2005. What is the average rate of increase for these two years?

Inscribed Similar Triangles

You may recall that if two objects are similar, corresponding angles are congruent and their sides are proportional in length. Let’s look at a right triangle, with an altitude drawn from the right angle.

There are three right triangles in this picture, $\triangle ADB$, $\triangle CDA$, and $\triangle CAB$. Both of the two smaller triangles are similar to the larger triangle because they each share an angle with $\triangle ADB$. That means all three triangles are similar to each other.

Theorem 8-5: If an altitude is drawn from the right angle of any right triangle, then the two triangles formed are similar to the original triangle and all three triangles are similar to each other.

The proof of Theorem 8-5 is in the review questions.

Example 1: Write the similarity statement for the triangles below.
Solution: If \( m\angle E = 30^\circ \), then \( m\angle I = 60^\circ \) and \( m\angle TRE = 60^\circ \). \( m\angle IRT = 30^\circ \) because it is complementary to \( \angle TRE \).

Line up the congruent angles in the similarity statement. \( \triangle IRE \sim \triangle ITR \sim \triangle RTE \)

We can also use the side proportions to find the length of the altitude.

Example 2: Find the value of \( x \).

Example 3: Find the value of \( x \).
Solution: Let’s set up a proportion.

\[
\frac{\text{shorter leg in } \triangle SVT}{\text{shorter leg in } \triangle RST} = \frac{\text{hypotenuse in } \triangle SVT}{\text{hypotenuse in } \triangle RST}
\]

\[
\frac{4}{x} = \frac{x}{20}
\]

\[
x^2 = 80
\]

\[
x = \sqrt{80} = 4\sqrt{5}
\]

Example 4: Find the value of \(y\) in \(\triangle RST\) above.

Solution: Use the Pythagorean Theorem.

\[
y^2 + (4\sqrt{5})^2 = 20^2
\]

\[
y^2 + 80 = 400
\]

\[
y^2 = 320
\]

\[
y = \sqrt{320} = 8\sqrt{5}
\]

The Geometric Mean

You are probably familiar with the arithmetic mean, which divides the sum of \(n\) numbers by \(n\). This is commonly used to determine the average test score for a group of students.

The geometric mean is a different sort of average, which takes the \(n^{th}\) root of the product of \(n\) numbers. In this text, we will primarily compare two numbers, so we would be taking the square root of the product of two numbers. This mean is commonly used with rates of increase or decrease.

Geometric Mean: The geometric mean of two positive numbers \(a\) and \(b\) is the number \(x\), such that \(\frac{a}{x} = \frac{x}{b}\) or \(x^2 = ab\) and \(x = \sqrt{ab}\).

Example 5: Find the geometric mean of 24 and 36.

Solution: \(x = \sqrt{24 \cdot 36} = \sqrt{12 \cdot 2 \cdot 12 \cdot 3} = 12\sqrt{6}\)

Example 6: Find the geometric mean of 18 and 54.

Solution: \(x = \sqrt{18 \cdot 54} = \sqrt{18 \cdot 18 \cdot 3} = 18\sqrt{3}\)

Notice that in both of these examples, we did not actually multiply the two numbers together, but kept them separate. This made it easier to simplify the radical.
A practical application of the geometric mean is to find the altitude of a right triangle.

**Example 7:** Find the value of $x$.

![Diagram of right triangle with altitude]

**Solution:** Using similar triangles, we have the proportion

\[
\frac{\text{shortest leg of smallest } \triangle}{\text{shortest leg of middle } \triangle} = \frac{\text{longer leg of smallest } \triangle}{\text{longer leg of middle } \triangle}
\]

\[
\frac{9}{x} = \frac{x}{27}
\]

\[
x^2 = 243
\]

\[
x = \sqrt{243} = 9\sqrt{3}
\]

In Example 7, $\frac{9}{x} = \frac{x}{27}$ is in the definition of the geometric mean. So, the altitude is the geometric mean of the two segments that it divides the hypotenuse into.

**Theorem 8-6:** In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of the altitude is the geometric mean of these two segments.

**Theorem 8-7:** In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of each leg of the right triangle is the geometric mean of the lengths of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.

![Diagram of right triangle with altitude]

In other words

**Theorem 8-6:** $\frac{BC}{AC} = \frac{AC}{DC}$

**Theorem 8-7:** $\frac{BC}{AB} = \frac{AB}{DB}$ and $\frac{DC}{AD} = \frac{AD}{DB}$

Both of these theorems are proved using similar triangles.

**Example 8:** Find the value of $x$ and $y$. 
Solution: Use theorem 8-7 to solve for $x$ and $y$.

\[
\frac{20}{x} = \frac{35}{y} \\
x^2 = 20 \cdot 35 \\
x = \sqrt{20 \cdot 35} \\
x = 10 \sqrt{7}
\]

\[
\frac{15}{y} = \frac{35}{y} \\
y^2 = 15 \cdot 35 \\
y = \sqrt{15 \cdot 35} \\
y = 5 \sqrt{21}
\]

You could also use the Pythagorean Theorem to solve for $y$, once $x$ has been solved for.

\[
(10 \sqrt{7})^2 + y^2 = 35^2 \\
700 + y^2 = 1225 \\
y = \sqrt{525} = 5 \sqrt{21}
\]

Either method is acceptable.

**Know What? Revisited** The average rate of increase can be found by using the geometric mean.

\[
x = \sqrt{0.213 \cdot 0.16} = 0.1846
\]

Over the two year period, housing prices increased 18.46%.

**Review Questions**

Use the diagram to answer questions 1-4.
1. Write the similarity statement for the three triangles in the diagram.
2. If $JM = 12$ and $ML = 9$, find $KM$.
3. Find $JK$.
4. Find $KL$.

Find the geometric mean between the following two numbers. Simplify all radicals.

5. 16 and 32
6. 45 and 35
7. 10 and 14
8. 28 and 42
9. 40 and 100
10. 51 and 8

Find the length of the missing variable(s). Simplify all radicals.
20. Write a proof for Theorem 8-5.

\[ \text{Given: } \triangle ABD \text{ with } \overline{AC} \perp \overline{DB} \text{ and } \angle DAB \text{ is a right angle. Prove: } \triangle ABD \sim \triangle CBA \sim \triangle CAD \]

21. Fill in the blanks for the proof of Theorem 8-7.

\[ \text{Given: } \triangle ABD \text{ with } \overline{AC} \perp \overline{DB} \text{ and } \angle DAB \text{ is a right angle. Prove: } \frac{BC}{AB} = \frac{AB}{DB} \]

**Table 6.3:**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \triangle ABD \text{ with } \overline{AC} \perp \overline{DB} \text{ and } \angle DAB \text{ is a right angle.} )</td>
<td></td>
</tr>
<tr>
<td>2. ( \triangle ABD \sim \triangle CBA \sim \triangle CAD )</td>
<td></td>
</tr>
<tr>
<td>3. ( \frac{BC}{AB} = \frac{AB}{DB} )</td>
<td></td>
</tr>
</tbody>
</table>

22. Last year Poorva’s rent increased by 5% and this year her landlord wanted to raise her rent by 7.5%. What is the average rate at which her landlord has raised her rent over the course of these two years?
23. Mrs. Smith teaches AP Calculus. Between the first and second years she taught the course her students’ average score improved by 12%. Between the second and third years, the scores increased by 9%. What is the average rate of improvement in her students’ scores?

24. According to the US Census Bureau, the rate of growth of the US population was 0.8% and in 2009 it was 1.0%. What was the average rate of population growth during that time period?

Algebra Connection

A geometric sequence is a sequence of numbers in which each successive term is determined by multiplying the previous term by the common ratio. An example is the sequence 1, 3, 9, 27, ... Here each term is multiplied by 3 to get the next term in the sequence. Another way to look at this sequence is to compare the ratios of the consecutive terms.

25. Find the ratio of the 2nd to 1st terms and the ratio of the 3rd to 2nd terms. What do you notice? Is this true for the next set (4th to 3rd terms)?

26. Given the sequence 4, 8, 16,..., if we equate the ratios of the consecutive terms we get: \[
\frac{8}{4} = \frac{16}{8}.
\]
This means that 8 is the __________________ of 4 and 16. We can generalize this to say that every term in a geometric sequence is the _________________ of the previous and subsequent terms.

Use what you discovered in problem 26 to find the middle term in the following geometric sequences.

27. 5, ____, 20
28. 4, ____, 100
29. 2, ____,
30. \(\frac{1}{2}\)

30. We can use what we have learned in this section in another proof of the Pythagorean Theorem. Use the diagram to fill in the blanks in the proof below.

![Diagram of a right triangle with labels A, B, C, a, b, c, d, e, f, and k.]

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (\frac{e}{a} = \frac{1}{4}) and (\frac{d}{b} = \frac{1}{2})</td>
<td>Theorem 8-7</td>
</tr>
<tr>
<td>2. (a^2 = e(d + e)) and (b^2 = d(e + e))</td>
<td>?</td>
</tr>
<tr>
<td>3. (a^2 + b^2 =?)</td>
<td>Combine equations from #2.</td>
</tr>
<tr>
<td>4. ?</td>
<td>Distributive Property</td>
</tr>
<tr>
<td>5. (c = d + e)</td>
<td>?</td>
</tr>
<tr>
<td>6. ?</td>
<td>Substitution PoE</td>
</tr>
</tbody>
</table>

**Table 6.4:**

**Review Queue Answers**

1. No, another angle besides the right angles must also be congruent.
2. Yes, the three angles in an isosceles right triangle are 45°, 45°, and 90°. Isosceles right triangles will always be similar.

3. \( \frac{3}{x} = \frac{x}{27} \rightarrow x^2 = 81 \rightarrow x = \pm 9 \)

4. \( 4^2 + 4^2 = h^2 \)
   \[ h = \sqrt{32} = 4\sqrt{2} \]
6.4 Tangent, Sine and Cosine

Learning Objectives

- Use the tangent, sine and cosine ratios in a right triangle.
- Understand these trigonometric ratios in special right triangles.
- Use a scientific calculator to find sine, cosine and tangent.
- Use trigonometric ratios in real-life situations.

Review Queue

1. The legs of an isosceles right triangle have length 14. What is the hypotenuse?
2. Do the lengths 8, 16, 20 make a right triangle? If not, is the triangle obtuse or acute?
3. In a 30-60-90 triangle, what do the 30, 60, and 90 refer to?
4. Find the measure of the missing lengths.

Know What? A restaurant needs to build a wheelchair ramp for its customers. The angle of elevation for a ramp is recommended to be $5^\circ$. If the vertical distance from the sidewalk to the front door is two feet, what is the horizontal distance that the ramp will take up ($x$)? How long will the ramp be ($y$)? Round your answers to the nearest hundredth.
6.4. Tangent, Sine and Cosine

What is Trigonometry?

The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define three trigonometric (or trig) functions. Once we have defined these functions, we will be able to solve problems like the Know What? above.

Trigonometry: The study of the relationships between the sides and angles of right triangles.

In trigonometry, sides are named in reference to a particular angle. The hypotenuse of a triangle is always the same, but the terms *adjacent* and *opposite* depend on which angle you are referencing. A side adjacent to an angle is the leg of the triangle that helps form the angle. A side opposite to an angle is the leg of the triangle that does not help form the angle. We never reference the right angle when referring to trig ratios.

\[ a \text{ is adjacent to } \angle B. \quad a \text{ is opposite } \angle A. \]

\[ b \text{ is adjacent to } \angle A. \quad b \text{ is opposite } \angle B. \]

\[ c \text{ is the hypotenuse.} \]

Sine, Cosine, and Tangent Ratios

The three basic trig ratios are called, sine, cosine and tangent. At this point, we will only take the sine, cosine and tangent of acute angles. However, you will learn that you can use these ratios with obtuse angles as well.

**Sine Ratio:** For an acute angle \( x \) in a right triangle, the \( \sin x \) is equal to the ratio of the side opposite the angle over the hypotenuse of the triangle.

Using the triangle above, \( \sin A = \frac{a}{c} \) and \( \sin B = \frac{b}{c} \).

**Cosine Ratio:** For an acute angle \( x \) in a right triangle, the \( \cos x \) is equal to the ratio of the side adjacent to the angle over the hypotenuse of the triangle.

Using the triangle above, \( \cos A = \frac{b}{c} \) and \( \cos B = \frac{a}{c} \).

**Tangent Ratio:** For an acute angle \( x \), in a right triangle, the \( \tan x \) is equal to the ratio of the side opposite to the angle over the side adjacent to \( x \).

Using the triangle above, \( \tan A = \frac{a}{b} \) and \( \tan B = \frac{b}{a} \).

There are a few important things to note about the way we write these ratios. First, keep in mind that the abbreviations \( \sin x \), \( \cos x \), and \( \tan x \) are all functions. Each ratio can be considered a function of the angle (see Chapter 10). Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write \( \sin x \) it is still pronounced *sine*, with a long “i”. When we write \( \cos x \), we still say co-sine. And when we write \( \tan x \), we still say tangent.

An easy way to remember ratios is to use the pneumonic SOH-CAH-TOA.
Example 1: Find the sine, cosine and tangent ratios of $\angle A$.

Solution: First, we need to use the Pythagorean Theorem to find the length of the hypotenuse.

\[
5^2 + 12^2 = h^2
\]
\[
13 = h
\]

So, $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, and $\tan A = \frac{12}{5}$.

A few important points:

- Always reduce ratios when you can.
- Use the Pythagorean Theorem to find the missing side (if there is one).
- The tangent ratio can be bigger than 1 (the other two cannot).
- If two right triangles are similar, then their sine, cosine, and tangent ratios will be the same (because they will reduce to the same ratio).
- If there is a radical in the denominator, rationalize the denominator.

Example 2: Find the sine, cosine, and tangent of $\angle B$.

Solution: Find the length of the missing side.

\[
AC^2 + 5^2 = 15^2
\]
\[
AC = 10 \sqrt{2}
\]

Therefore, $\sin B = \frac{10 \sqrt{2}}{15} = \frac{2 \sqrt{2}}{3}$, $\cos B = \frac{5}{15} = \frac{1}{3}$, and $\tan B = \frac{10 \sqrt{2}}{5} = 2 \sqrt{2}$.

Example 3: Find the sine, cosine and tangent of $30^\circ$. 

\[
\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}
\]
6.4. Tangent, Sine and Cosine

Solution: This is a special right triangle, a 30-60-90 triangle. So, if the short leg is 6, then the long leg is \( 6 \sqrt{3} \) and the hypotenuse is 12.

\[
\sin 30^\circ = \frac{6}{12} = \frac{1}{2}, \cos 30^\circ = \frac{6 \sqrt{3}}{12} = \frac{\sqrt{3}}{2}, \text{ and } \tan 30^\circ = \frac{6}{6 \sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}.
\]

In Example 3, we knew the angle measure of the angle we were taking the sine, cosine and tangent of. This means that the sine, cosine and tangent for an angle are fixed.

Sine, Cosine, and Tangent with a Calculator

We now know that the trigonometric ratios are not dependent on the sides, but the ratios. Therefore, there is one fixed value for every angle, from 0\(^\circ\) to 90\(^\circ\). Your scientific (or graphing) calculator knows the values of the sine, cosine and tangent of all of these angles. Depending on your calculator, you should have [SIN], [COS], and [TAN] buttons. Use these to find the sine, cosine, and tangent of any acute angle.

Example 4: Find the indicated trigonometric value, using your calculator.

a) \( \sin 78^\circ \)
b) \( \cos 60^\circ \)
c) \( \tan 15^\circ \)

Solution: Depending on your calculator, you enter the degree first, and then press the correct trig button or the other way around. For TI-83s and TI-84s you press the trig button first, followed by the angle. Also, make sure the mode of your calculator is in DEGREES.

a) \( \sin 78^\circ = 0.9781 \)
b) \( \cos 60^\circ = 0.5 \)
c) \( \tan 15^\circ = 0.2679 \)

Finding the Sides of a Triangle using Trig Ratios

One application of the trigonometric ratios is to use them to find the missing sides of a right triangle. All you need is one angle, other than the right angle, and one side. Let’s go through a couple of examples.

Example 5: Find the value of each variable. Round your answer to the nearest hundredth.
Solution: We are given the hypotenuse, so we would need to use the sine to find $b$, because it is opposite $22^\circ$ and cosine to find $a$, because it is adjacent to $22^\circ$.

\[
\sin 22^\circ = \frac{b}{30} \quad \cos 22^\circ = \frac{a}{30}
\]

\[
30 \cdot \sin 22^\circ = b \quad 30 \cdot \cos 22^\circ = a
\]

\[
b \approx 11.24 \\
a \approx 27.82
\]

Example 6: Find the value of each variable. Round your answer to the nearest hundredth.

\[
\begin{align*}
\cos 42^\circ &= \frac{9}{c} \\
9 \cdot \cos 42^\circ &= 9 \\
c &= \frac{9}{\cos 42^\circ} \approx 12.11
\end{align*}
\]

\[
\tan 42^\circ = \frac{d}{9} \quad 9 \cdot \tan 42^\circ = d
\]

\[
d \approx 8.10
\]

Notice in both of these examples, you should only use the information that you are given. For example, you should not use the found value of $b$ to find $a$ (in Example 5) because $b$ is an approximation. Use exact values to give the most accurate answers. However, in both examples you could have also used the complementary angle to the one given.

Angles of Depression and Elevation

Another practical application of the trigonometric functions is to find the measure of lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

Angle of Depression: The angle measured from the horizon or horizontal line, down.
Angle of Elevation: The angle measure from the horizon or horizontal line, up.

Example 7: An inquisitive math student is standing 25 feet from the base of the Washington Monument. The angle of elevation from her horizontal line of sight is 87.4°. If her “eye height” is 5ft, how tall is the monument?

Solution: We can find the height of the monument by using the tangent ratio and then adding the eye height of the student.

\[
\tan 87.4° = \frac{h}{25}
\]

\[
h = 25 \cdot \tan 87.4° = 550.54
\]

Adding 5 ft, the total height of the Washington Monument is 555.54 ft.

According to Wikipedia, the actual height of the monument is 555.427 ft.

Know What? Revisited To find the horizontal length and the actual length of the ramp, we need to use the tangent and sine.

\[
\tan 5° = \frac{2}{x} \quad \quad \quad \sin 5° = \frac{2}{y}
\]

\[
x = \frac{2}{\tan 5°} = 22.86
\]

\[
y = \frac{2}{\sin 5°} = 22.95
\]

Review Questions

Use the diagram to fill in the blanks below.
1. \( \tan D = \frac{?}{?} \)
2. \( \sin F = \frac{?}{?} \)
3. \( \tan F = \frac{?}{?} \)
4. \( \cos F = \frac{?}{?} \)
5. \( \sin D = \frac{?}{?} \)
6. \( \cos D = \frac{?}{?} \)

From questions 1-6, we can conclude the following. Fill in the blanks.

7. \( \cos \underline{\text{__} \quad \text{__}} = \sin F \) and \( \sin \underline{\text{__} \quad \text{__}} = \cos F \)
8. The sine of an angle is \underline{\text{__} \quad \text{__}} to the cosine of its \underline{\text{__} \quad \text{__}}.
9. \( \tan D \) and \( \tan F \) are \underline{\text{__} \quad \text{__}} of each other.

Use your calculator to find the value of each trig function below. Round to four decimal places.

10. \( \sin 24^\circ \)
11. \( \cos 45^\circ \)
12. \( \tan 88^\circ \)
13. \( \sin 43^\circ \)

Find the sine, cosine and tangent of \( \angle A \). Reduce all fractions and radicals.

Find the length of the missing sides. Round your answers to the nearest hundredth.
23. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point, is 250 feet. How far away is she from the reef?

24. The Leaning Tower of Piza currently “leans” at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?
25. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 72°. If the building is 78 ft tall, how far away is the fountain?

26. William spots a tree directly across the river from where he is standing. He then walks 20 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 65°. How wide is the river?

27. Diego is flying his kite one afternoon and notices that he has let out the entire 120 ft of string. The angle his string makes with the ground is 52°. How high is his kite at this time?

28. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 36° angle with the ground 25 ft from the base of the tree. What was the height of the tree to the nearest foot?

29. Upon descent an airplane is 20,000 ft above the ground. The air traffic control tower is 200 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.

30. **Critical Thinking** Why are the sine and cosine ratios always be less than 1?

---

**Review Queue Answers**

1. The hypotenuse is $14\sqrt{2}$.
2. No, $8^2 + 16^2 < 20^2$, the triangle is obtuse.
3. 30°, 60°, and 90° refer to the angle measures in the special right triangle.
4. Answers:
   a. $x = 2, y = 2\sqrt{3}$
   b. $x = 6\sqrt{3}, y = 18, z = 18\sqrt{3}, w = 36$
6.5 Inverse Trigonometric Ratios

Learning Objectives

- Use the inverse trigonometric ratios to find an angle in a right triangle.
- Solve a right triangle.
- Apply inverse trigonometric ratios to real-life situation and special right triangles.

Review Queue

Find the lengths of the missing sides. Round your answer to the nearest hundredth.

1. 
2. 
3. Draw an isosceles right triangle with legs of length 3. What is the hypotenuse?
4. Use the triangle from #3, to find the sine, cosine, and tangent of 45°.
5. Explain why \( \tan 45° = 1 \).

**Know What?** The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115 ft high. What is the angle of elevation, \( x° \), of this escalator?
Inverse Trigonometric Ratios

The word *inverse* is probably familiar to you. In mathematics, once you learn how to do an operation, you also learn how to “undo” it. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In algebra you used inverse operations to solve equations and inequalities.

When we apply the word inverse to the trigonometric ratios, we can find the acute angle measures within a right triangle. Normally, if you are given an angle and a side of a right triangle, you can find the other two sides, using sine, cosine or tangent. With the inverse trig ratios, you can find the angle measure, given two sides.

**Inverse Tangent:** If you know the opposite side and adjacent side of an angle in a right triangle, you can use inverse tangent to find the measure of the angle.

Inverse tangent is also called arctangent and is labeled \( \tan^{-1} \) or \( \arctan \). The “-1” indicates inverse.

**Inverse Sine:** If you know the opposite side of an angle and the hypotenuse in a right triangle, you can use inverse sine to find the measure of the angle.

Inverse sine is also called arcsine and is labeled \( \sin^{-1} \) or \( \arcsin \).

**Inverse Cosine:** If you know the adjacent side of an angle and the hypotenuse in a right triangle, you can use inverse cosine to find the measure of the angle.

Inverse cosine is also called arccosine and is labeled \( \cos^{-1} \) or \( \arccos \).

Using the triangle below, the inverse trigonometric ratios look like this:

![Triangle with labeled sides and angles](image)

\[
\begin{align*}
\tan^{-1} \left( \frac{b}{a} \right) &= m_{\angle B} & \tan^{-1} \left( \frac{a}{b} \right) &= m_{\angle A} \\
\sin^{-1} \left( \frac{b}{c} \right) &= m_{\angle B} & \sin^{-1} \left( \frac{a}{c} \right) &= m_{\angle A} \\
\cos^{-1} \left( \frac{a}{c} \right) &= m_{\angle B} & \cos^{-1} \left( \frac{b}{c} \right) &= m_{\angle A}
\end{align*}
\]

In order to actually find the measure of the angles, you will need you use your calculator. On most scientific and graphing calculators, the buttons look like \([\text{SIN}^{-1}],[\text{COS}^{-1}]\), and \([\text{TAN}^{-1}]\). Typically, you might have to hit a shift or 2\(^{nd}\) button to access these functions. For example, on the TI-83 and 84, \([2^{nd}][\text{SIN}]\) is \([\text{SIN}^{-1}]\). Again, make sure the mode is in degrees.

When you find the inverse of a trigonometric function, you put the word *arc* in front of it. So, the inverse of a tangent is called the arctangent (or arctan for short). Think of the arctangent as a tool you can use like any other inverse operation when solving a problem. If tangent tells you the ratio of the lengths of the sides opposite and adjacent to an angle, then tangent inverse tells you the measure of an angle with a given ratio.

**Example 1:** Use the sides of the triangle and your calculator to find the value of \( \angle A \). Round your answer to the nearest tenth of a degree.
Solution: In reference to $\angle A$, we are given the opposite leg and the adjacent leg. This means we should use the tangent ratio.

$$\tan A = \frac{20}{25} = \frac{4}{5}, \text{ therefore } \tan^{-1} \left( \frac{4}{5} \right) = m \angle A.$$ Use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: $[2^{nd}] [\text{TAN}] \left( \frac{4}{5} \right) \text{[ENTER]}$ and the screen looks like:

![Inverse tangent calculation]

So, $m \angle A = 38.7^\circ$

Example 2: $\angle A$ is an acute angle in a right triangle. Use your calculator to find $m \angle A$ to the nearest tenth of a degree.

a) $\sin A = 0.68$

b) $\cos A = 0.85$

c) $\tan A = 0.34$

Solution:

a) $m \angle A = \sin^{-1} 0.68 = 42.8^\circ$

b) $m \angle A = \cos^{-1} 0.85 = 31.8^\circ$

c) $m \angle A = \tan^{-1} 0.34 = 18.8^\circ$

Solving Triangles

Now that we know how to use inverse trigonometric ratios to find the measure of the acute angles in a right triangle, we can solve right triangles. To solve a right triangle, you would need to find all sides and angles in a right triangle, using any method. When solving a right triangle, you could use sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem. Remember when solving right triangles to only use the values that you are given.

Example 3: Solve the right triangle.

Solution: To solve this right triangle, we need to find $AB, m \angle C$ and $m \angle B$. Use $AC$ and $CB$ to give the most accurate answers.
**AB:** Use the Pythagorean Theorem.

\[24^2 + AB^2 = 30^2\]
\[576 + AB^2 = 900\]
\[AB^2 = 324\]
\[AB = \sqrt{324} = 18\ m\]

**m B:** Use the inverse sine ratio.

\[\sin B = \frac{24}{30} = \frac{4}{5}\]
\[\sin^{-1} \left( \frac{4}{5} \right) = 53.1^\circ = m \angle B\]

**m C:** Use the inverse cosine ratio.

\[\cos C = \frac{24}{30} = \frac{4}{5}\]
\[\cos^{-1} \left( \frac{4}{5} \right) = 36.9^\circ = m \angle C\]

**Example 4:** Solve the right triangle.

**Solution:** To solve this right triangle, we need to find \(AB, BC\) and \(m \angle A\).

**AB:** Use sine ratio.

\[\sin 62^\circ = \frac{25}{AB}\]
\[AB = \frac{25}{\sin 62^\circ}\]
\[AB \approx 28.31\]

**BC:** Use tangent ratio.

\[\tan 62^\circ = \frac{25}{BC}\]
\[BC = \frac{25}{\tan 62^\circ}\]
\[BC \approx 13.30\]
6.5. Inverse Trigonometric Ratios

\[ m \angle A: \text{ Use Triangle Sum Theorem} \]

\[ 62^\circ + 90^\circ + m \angle A = 180^\circ \]
\[ m \angle A = 28^\circ \]

**Example 5:** Solve the right triangle.

![Diagram of a right triangle with sides 15 and angles labeled]

**Solution:** Even though, there are no angle measures given, we know that the two acute angles are congruent, making them both 45°. Therefore, this is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

**Trigonometric Ratios**

\[
\tan 45^\circ = \frac{BC}{15} \quad \quad \sin 45^\circ = \frac{AC}{15}
\]
\[
BC = \frac{15}{\tan 45^\circ} = 15 \quad \quad AC = \frac{15}{\sin 45^\circ} \approx 21.21
\]

**45-45-90 Triangle Ratios**

\[ BC = AB = 15, AC = 15\sqrt{2} \approx 21.21 \]

**Real-Life Situations**

Much like the trigonometric ratios, the inverse trig ratios can be used in several real-life situations. Here are a couple examples.

**Example 6:** A 25 foot tall flagpole casts a 42 feet shadow. What is the angle that the sun hits the flagpole?
Solution: First, draw a picture. The angle that the sun hits the flagpole is the acute angle at the top of the triangle, \( x^\circ \). From the picture, we can see that we need to use the inverse tangent ratio.

\[
\tan x = \frac{42}{25} \\
\tan^{-1} \left( \frac{42}{25} \right) \approx 59.2^\circ = x
\]

Example 7: Elise is standing on the top of a 50 foot building and spots her friend, Molly across the street. If Molly is 35 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise’s eye height is 4.5 feet.

Solution: Because of parallel lines, the angle of depression is equal to the angle at Molly, or \( x^\circ \). We can use the inverse tangent ratio.

\[
\tan^{-1} \left( \frac{54.5}{30} \right) = 61.2^\circ = x
\]

Know What? Revisited To find the escalator’s angle of elevation, we need to use the inverse sine ratio.

\[
\sin^{-1} \left( \frac{115}{230} \right) = 30^\circ \quad \text{The angle of elevation is } 30^\circ.
\]

Review Questions

Use your calculator to find \( m\angle A \) to the nearest tenth of a degree.
Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$

Solving the following right triangles. Find all missing sides and angles.
16. **Writing** Explain when to use a trigonometric ratio to find a side length of a right triangle and when to use the Pythagorean Theorem.

**Real-Life Situations** Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

17. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
18. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?
19. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?
20. Elizabeth wants to know the angle at which the sun hits a tree in her backyard at 3 pm. She finds that the length of the tree’s shadow is 24 ft at 3 pm. At the same time of day, her shadow is 6 ft 5 inches. If Elizabeth is 4 ft 8 inches tall, find the height of the tree and hence the angle at which the sunlight hits the tree.
21. Alayna is trying to determine the angle at which to aim her sprinkler nozzle to water the top of a 5 ft bush in her yard. Assuming the water takes a straight path and the sprinkler is on the ground 4 ft from the tree, at what angle of inclination should she set it?
22. **Science Connection** Would the answer to number 20 be the same every day of the year? What factors would influence this answer? How about the answer to number 21? What factors might influence the path of the water?
23. Tommy was solving the triangle below and made a mistake. What did he do wrong?

\[
\tan^{-1}\left(\frac{21}{28}\right) \approx 36.9^\circ
\]

24. Tommy then continued the problem and set up the equation: \(\cos 36.9^\circ = \frac{21}{x}\). By solving this equation he found that the hypotenuse was 26.3 units. Did he use the correct trigonometric ratio here? Is his answer correct? Why or why not?

25. How could Tommy have found the hypotenuse in the triangle another way and avoided making his mistake?

**Examining Patterns** Below is a table that shows the sine, cosine, and tangent values for eight different angle measures. Answer the following questions.

<table>
<thead>
<tr>
<th>Angle (°)</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine</td>
<td>0.1736</td>
<td>0.3420</td>
<td>0.5</td>
<td>0.6428</td>
<td>0.7660</td>
<td>0.8660</td>
<td>0.9397</td>
<td>0.9848</td>
</tr>
<tr>
<td>Cosine</td>
<td>0.9848</td>
<td>0.9397</td>
<td>0.8660</td>
<td>0.7660</td>
<td>0.6428</td>
<td>0.5</td>
<td>0.3420</td>
<td>0.1736</td>
</tr>
<tr>
<td>Tangent</td>
<td>0.1763</td>
<td>0.3640</td>
<td>0.5774</td>
<td>0.8391</td>
<td>1.1918</td>
<td>1.7321</td>
<td>2.7475</td>
<td>5.6713</td>
</tr>
</tbody>
</table>

26. What value is equal to \(\sin 40^\circ\)?
27. What value is equal to \(\cos 70^\circ\)?
28. Describe what happens to the sine values as the angle measures increase.
29. Describe what happens to the cosine values as the angle measures increase.
30. What two numbers are the sine and cosine values between?
31. Find \(\tan 85^\circ\), \(\tan 89^\circ\), and \(\tan 89.5^\circ\) using your calculator. Now, describe what happens to the tangent values as the angle measures increase.
32. Explain why all of the sine and cosine values are less than one. (hint: think about the sides in the triangle and the relationships between their lengths)

**Review Queue Answers**

1. \(\sin 36^\circ = \frac{y}{7}\)  \(\cos 36^\circ = \frac{x}{7}\)  
   \(y = 4.11\)  \(x = 5.66\)
2. \(\cos 12.7^\circ = \frac{40}{x}\)  \(\tan 12.7^\circ = \frac{y}{40}\)  
   \(x = 41.00\)  \(y = 9.01\)
3. 

332
4. \[
\sin 45^\circ = \frac{\frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}}} = \frac{\sqrt{2}}{2}
\]
\[
\cos 45^\circ = \frac{\frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}}} = \frac{\sqrt{2}}{2}
\]
\[
\tan 45^\circ = \frac{\frac{3}{3}}{\frac{3}{3}} = 1
\]

5. The tangent of $45^\circ$ equals one because it is the ratio of the opposite side over the adjacent side. In an isosceles right triangle, or 45-45-90 triangle, the opposite and adjacent sides are the same, making the ratio always 1.
Keywords Theorems

- 45-45-90 Corollary
- 30-60-90 Corollary
- Trigonometry
- Adjacent (Leg)
- Opposite (Leg)
- Sine Ratio
- Cosine Ratio
- Tangent Ratio
- Angle of Depression
- Angle of Elevation
- Inverse Tangent
- Inverse Sine
- Inverse Cosine

Review Questions

Solve the following right triangles using the Pythagorean Theorem, the trigonometric ratios, and the inverse trigonometric ratios. When possible, simplify the radical. If not, round all decimal answers to the nearest tenth.