GEOMETRY SKILL BUILDERS
(Extra Practice)

Introduction to Students and Their Teachers

Learning is an individual endeavor. Some ideas come easily; others take time--sometimes lots of time--to grasp. In addition, individual students learn the same idea in different ways and at different rates. The authors of this textbook designed the classroom lessons and homework to give students time--often weeks and months--to practice an idea and to use it in various settings. This section of the textbook offers students a brief review of 27 topics followed by additional practice with answers. Not all students will need extra practice. Some will need to do a few topics, while others will need to do many of the sections to help develop their understanding of the ideas. This section of the text may also be useful to prepare for tests, especially final examinations.

How these problems are used will be up to your teacher, your parents, and yourself. In classes where a topic needs additional work by most students, your teacher may assign work from one of the skill builders that follow. In most cases, though, the authors expect that these resources will be used by individual students who need to do more than the textbook offers to learn an idea. This will mean that you are going to need to do some extra work outside of class. In the case where additional practice is necessary for you individually or for a few students in your class, you should not expect your teacher to spend time in class going over the solutions to the skill builder problems. After reading the examples and trying the problems, if you still are not successful, talk to your teacher about getting a tutor or extra help outside of class time.

Warning! Looking is not the same as doing. You will never become good at any sport just by watching it. In the same way, reading through the worked out examples and understanding the steps are not the same as being able to do the problems yourself. An athlete only gets good with practice. The same is true of developing your algebra and geometry skills. How many of the extra practice problems do you need to try? That is really up to you. Remember that your goal is to be able to do problems of the type you are practicing on your own, confidently and accurately.

Two other sources for help with the algebra and geometry topics in this course are the Parent's Guide with Review to Math 2 (Geometry) and the Parent's Guide with Review to Math 1 (Algebra 1). Information about ordering these supplements can be found inside the front page of the student text. These resources are also available free from the internet at www.cpm.org.

Online homework help is underwritten by CPM at www.hotmath.com.

Skill Builder Topics

1. Writing and graphing linear equations
2. The Pythagorean Theorem
3. Area
4. Solving linear systems
5. Properties of angles, lines, and triangles
6. Linear inequalities
7. Multiplying Polynomials
8. Right triangle trigonometry
9. Ratio of similarity
10. Similarity of length, area, and volume
11. Interior and exterior angles of polygons
12. Areas by dissection
13. Central and inscribed angles
14. Area of sectors
8. Factoring Polynomials
9. Zero Product Property and quadratics
10. The Quadratic Formula
11. Triangle Congruence
12. Laws of Exponents
13. Proof
14. Radicals
22. Tangents, secants, and chords
23. Volume and surface area of polyhedra
24. Simplifying rational expressions
25. Multiplication and division of rational expressions
26. Addition and subtraction of rational expressions
27. Solving mixed equations and inequalities
**WRITING AND GRAPHING LINEAR EQUATIONS ON A FLAT SURFACE #1**

**SLOPE** is a number that indicates the steepness (or flatness) of a line, as well as its direction (up or down) left to right.

**SLOPE** is determined by the ratio: \( \frac{\text{vertical change}}{\text{horizontal change}} \) between any two points on a line.

For lines that go **up** (from left to right), the sign of the slope is **positive**. For lines that go **down** (left to right), the sign of the slope is **negative**.

Any linear equation written as \( y = mx + b \), where \( m \) and \( b \) are any real numbers, is said to be in **SLOPE-INTERCEPT FORM**. \( m \) is the **SLOPE** of the line. \( b \) is the **Y-INTERCEPT**, that is, the point \((0, b)\) where the line intersects (crosses) the y-axis.

If two lines have the same slope, then they are parallel. Likewise, **PARALLEL LINES** have the same slope.

Two lines are **PERPENDICULAR** if the slope of one line is the negative reciprocal of the slope of the other line, that is, \( m \) and \( -\frac{1}{m} \). Note that \( m \cdot \left(-\frac{1}{m}\right) = -1 \).

Examples: \( \frac{3}{5} \) and \( -\frac{1}{3} \), \( -\frac{2}{3} \) and \( \frac{3}{2} \), \( \frac{5}{4} \) and \( -\frac{4}{5} \)

Two distinct lines on a flat surface that are not parallel intersect in a single point. See "Solving Linear Systems" to review how to find the point of intersection.

**Example 1**

Graph the linear equation \( y = \frac{4}{7} x + 2 \)

Using \( y = mx + b \), the slope in \( y = \frac{4}{7} x + 2 \) is \( \frac{4}{7} \) and the y-intercept is the point \((0, 2)\). To graph, begin at the y-intercept \((0, 2)\). Remember that slope is \( \frac{\text{vertical change}}{\text{horizontal change}} \) so go up 4 units (since 4 is positive) from \((0, 2)\) and then move right 7 units. This gives a second point on the graph. To create the graph, draw a straight line through the two points.
Example 2

A line has a slope of $\frac{3}{4}$ and passes through (3, 2). What is the equation of the line?

Using $y = mx + b$, write $y = \frac{3}{4}x + b$. Since (3, 2) represents a point (x, y) on the line, substitute 3 for $x$ and 2 for $y$, $2 = \frac{3}{4}(3) + b$, and solve for $b$.

$2 = \frac{9}{4} + b \Rightarrow 2 - \frac{9}{4} = b \Rightarrow -\frac{1}{4} = b$. The equation is $y = \frac{3}{4}x - \frac{1}{4}$.

Example 3

Decide whether the two lines at right are parallel, perpendicular, or neither (i.e., intersecting).

First find the slope of each equation. Then compare the slopes.

\[
\begin{align*}
5x - 4y &= -6 \\
-4y &= -5x - 6 \\
y &= \frac{5x + 6}{4} \\
y &= \frac{5}{4}x + \frac{3}{2} \\
\text{The slope of this line is } \frac{5}{4}.
\end{align*}
\]

\[
\begin{align*}
-4x + 5y &= 3 \\
5y &= 4x + 3 \\
y &= \frac{4x + 3}{5} \\
y &= \frac{4}{5}x + \frac{3}{5} \\
\text{The slope of this line is } \frac{4}{5}.
\end{align*}
\]

These two slopes are not equal, so they are not parallel. The product of the two slopes is 1, not -1, so they are not perpendicular. These two lines are neither parallel nor perpendicular, but do intersect.

Example 4

Find two equations of the line through the given point, one parallel and one perpendicular to the given line: $y = -\frac{5}{2}x + 5$ and (-4, 5).

For the parallel line, use $y = mx + b$ with the same slope to write $y = -\frac{5}{2}x + b$.

Substitute the point (-4, 5) for $x$ and $y$ and solve for $b$.

\[
5 = -\frac{5}{2}(-4) + b \Rightarrow 5 = \frac{20}{2} + b \Rightarrow 5 = b
\]

Therefore the parallel line through (-4, 5) is $y = -\frac{5}{2}x - 5$.

For the perpendicular line, use $y = mx + b$ where $m$ is the negative reciprocal of the slope of the original equation to write $y = \frac{2}{5}x + b$.

Substitute the point (-4, 5) and solve for $b$.

\[
5 = \frac{2}{5}(-4) + b \Rightarrow -\frac{8}{5} + b = b
\]

Therefore the perpendicular line through (-4, 5) is $y = \frac{2}{5}x + \frac{33}{5}$.
Identify the y-intercept in each equation.

1. \( y = \frac{1}{2} x - 2 \)
2. \( y = -\frac{3}{5} x - \frac{5}{3} \)
3. \( 3x + 2y = 12 \)
4. \( x - y = -13 \)
5. \( 2x - 4y = 12 \)
6. \( 4y - 2x = 12 \)

Write the equation of the line with:

7. slope = \( \frac{1}{2} \) and passing through (4, 3).
8. slope = \( \frac{2}{3} \) and passing through (-3, -2).
9. slope = \( -\frac{1}{3} \) and passing through (4, -1).
10. slope = -4 and passing through (-3, 5).

Determine the slope of each line using the highlighted points.

11.
12.
13.

Using the slope and y-intercept, determine the equation of the line.

14.
15.
16.
17.

Graph the following linear equations on graph paper.

18. \( y = \frac{1}{2} x + 3 \)
19. \( y = -\frac{3}{5} x - 1 \)
20. \( y = 4x \)
21. \( y = -6x + \frac{1}{2} \)
22. \( 3x + 2y = 12 \)
State whether each pair of lines is parallel, perpendicular, or intersecting.

23. \(y = 2x - 2\) and \(y = 2x + 4\)

24. \(y = \frac{1}{2}x + 3\) and \(y = -2x - 4\)

25. \(x - y = 2\) and \(x + y = 3\)

26. \(y - x = -1\) and \(y + x = 3\)

27. \(x + 3y = 6\) and \(y = \frac{1}{3}x - 3\)

28. \(3x + 2y = 6\) and \(2x + 3y = 6\)

29. \(4x = 5y - 3\) and \(4y = 5x + 3\)

30. \(3x - 4y = 12\) and \(4y = 3x + 7\)

Find an equation of the line through the given point and parallel to the given line.

31. \(y = 2x - 2\) and \((-3, 5)\)

32. \(y = \frac{1}{2}x + 3\) and \((-4, 2)\)

33. \(x - y = 2\) and \((-2, 3)\)

34. \(y - x = -1\) and \((-2, 1)\)

35. \(x + 3y = 6\) and \((-1, 1)\)

36. \(3x + 2y = 6\) and \((2, -1)\)

37. \(4x = 5y - 3\) and \((1, -1)\)

38. \(3x - 4y = 12\) and \((4, -2)\)

Find an equation of the line through the given point and perpendicular to the given line.

39. \(y = 2x - 2\) and \((-3, 5)\)

40. \(y = \frac{1}{2}x + 3\) and \((-4, 2)\)

41. \(x - y = 2\) and \((-2, 3)\)

42. \(y - x = -1\) and \((-2, 1)\)

43. \(x + 3y = 6\) and \((-1, 1)\)

44. \(3x + 2y = 6\) and \((2, -1)\)

45. \(4x = 5y - 3\) and \((1, -1)\)

46. \(3x - 4y = 12\) and \((4, -2)\)

Write an equation of the line parallel to each line below through the given point

47. \(y = 2x - 2\) and \((-3, 2)\)

48. \(x - y = 2\) and \((-2, 3)\)

49. Write the equation of the line through \((7, -8)\) which is parallel to the line through \((2, 5)\) and \((8, -3)\)

50. Write the equation of the line through \((1, -4)\) which is parallel to the line through \((-3, -7)\) and \((4, 3)\)
Answers

1. (0, –2)  
2. \( \left( 0, -\frac{5}{3} \right) \)  
3. (0, 6)  
4. (0, 13)

5. (0, –3)  
6. (0, 3)

7. \( y = \frac{1}{5} x + 1 \)  
8. \( y = \frac{2}{3} x \)

9. \( y = -\frac{1}{3} x + \frac{1}{3} \)  
10. \( y = -4x - 7 \)

11. \( -\frac{1}{2} \)  
12. \( \frac{3}{4} \)

13. –2  
14. \( y = 2x - 2 \)

15. \( y = -x + 2 \)  
16. \( y = \frac{1}{3} x + 2 \)

17. \( y = -2x + 4 \)  
18. line with slope \( \frac{1}{2} \) and y-intercept (0, 3)

19. line with slope \( -\frac{3}{5} \) and y-intercept (0, –1)  
20. line with slope 4 and y-intercept (0, 0)

21. line with slope \( -6 \) and y-intercept \( \left( 0, \frac{1}{2} \right) \)  
22. line with slope \( -\frac{3}{5} \) and y-intercept (0, 6)

23. parallel  
24. perpendicular  
25. perpendicular  
26. perpendicular

27. parallel  
28. intersecting  
29. intersecting  
30. parallel

31. \( y = 2x + 11 \)  
32. \( y = \frac{1}{2} x + 4 \)  
33. \( y = x + 5 \)  
34. \( y = x + 3 \)

35. \( y = -\frac{1}{3} x + \frac{2}{3} \)  
36. \( y = -\frac{3}{2} x + 2 \)  
37. \( y = \frac{4}{3} x - \frac{9}{5} \)  
38. \( y = \frac{3}{4} x - 5 \)

39. \( y = -\frac{1}{2} x + \frac{7}{2} \)  
40. \( y = -2x - 6 \)  
41. \( y = -x + 1 \)  
42. \( y = -x - 1 \)

43. \( y = 3x + 4 \)  
44. \( y = \frac{2}{3} x - \frac{7}{3} \)  
45. \( y = -\frac{5}{4} x + \frac{1}{4} \)  
46. \( y = -\frac{4}{5} x + \frac{10}{3} \)

47. \( y = 3x + 11 \)  
48. \( y = -\frac{1}{2} x + \frac{15}{2} \)  
49. \( y = -\frac{4}{3} x + \frac{4}{3} \)  
50. \( y = \frac{10}{7} x - \frac{38}{7} \)
PYTHAGOREAN THEOREM #2

Any triangle that has a right angle is called a **RIGHT TRIANGLE**. The two sides that form the right angle, a and b, are called **LEGS**, and the side opposite (that is, across the triangle from) the right angle, c, is called the **HYPOTENUSE**.

For any right triangle, the sum of the squares of the legs of the triangle is equal to the square of the hypotenuse, that is, \(a^2 + b^2 = c^2\). This relationship is known as the **PYTHAGOREAN THEOREM**. In words, the theorem states that:

\[
(\text{leg})^2 + (\text{leg})^2 = (\text{hypotenuse})^2.
\]

**Example**

Draw a diagram, then use the Pythagorean Theorem to write an equation or use area pictures (as shown on page 22, problem RC-1) on each side of the triangle to solve each problem.

a) Solve for the missing side.

\[
c^2 + 13^2 = 17^2
\]
\[
c^2 + 169 = 289
\]
\[
c^2 = 120
\]
\[
c = \sqrt{120}
\]
\[
c \approx 10.95
\]

b) Find x to the nearest tenth:

\[
(5x)^2 + x^2 = 20^2
\]
\[
25x^2 + x^2 = 400
\]
\[
26x^2 = 400
\]
\[
x^2 \approx 15.4
\]
\[
x \approx \sqrt{15.4}
\]
\[
x \approx 3.9
\]

c) One end of a ten foot ladder is four feet from the base of a wall. How high on the wall does the top of the ladder touch?

\[
x^2 + 4^2 = 10^2
\]
\[
x^2 + 16 = 100
\]
\[
x^2 = 84
\]
\[
x \approx 9.2
\]

The ladder touches the wall about 9.2 feet above the ground.

d) Could 3, 6 and 8 represent the lengths of the sides of a right triangle? Explain.

\[
3^2 + 6^2 \neq 8^2
\]
\[
9 + 36 \neq 64
\]
\[
45 \neq 64
\]

Since the Pythagorean Theorem relationship is not true for these lengths, they cannot be the side lengths of a right triangle.
Use the Pythagorean Theorem to find the value of $x$. Round answers to the nearest tenth.

1. 

\[
\begin{array}{c}
37 \\
22
\end{array}
\]

2. 

\[
\begin{array}{c}
20 \\
x
\end{array}
\]

3. 

\[
\begin{array}{c}
x \\
42
\end{array}
\]

4. 

\[
\begin{array}{c}
x \\
83
\end{array}
\]

5. 

\[
\begin{array}{c}
x \\
72
\end{array}
\]

6. 

\[
\begin{array}{c}
x \\
15
\end{array}
\]

7. 

\[
\begin{array}{c}
x \\
32
\end{array}
\]

8. 

\[
\begin{array}{c}
x \\
16
\end{array}
\]

9. 

\[
\begin{array}{c}
x \\
105
\end{array}
\]

10. 

\[
\begin{array}{c}
x \\
30
\end{array}
\]

Solve the following problems.

11. A 12 foot ladder is six feet from a wall. How high on the wall does the ladder touch?

12. A 15 foot ladder is five feet from a wall. How high on the wall does the ladder touch?

13. A 9 foot ladder is three feet from a wall. How high on the wall does the ladder touch?

14. A 12 foot ladder is three and a half feet from a wall. How high on the wall does the ladder touch?

15. A 6 foot ladder is one and a half feet from a wall. How high on the wall does the ladder touch?

16. Could 2, 3, and 6 represent the lengths of sides of a right angle triangle? Justify your answer.

17. Could 8, 12, and 13 represent the lengths of sides of a right triangle? Justify your answer.

18. Could 5, 12, and 13 represent the lengths of sides of a right triangle? Justify your answer.

19. Could 9, 12, and 15 represent the lengths of sides of a right triangle? Justify your answer.

20. Could 10, 15, and 20 represent the lengths of sides of a right triangle? Justify your answer.

Answers

1. 29.7 
2. 93.9 
3. 44.9 
4. 69.1 
5. 31.0
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<td>10</td>
<td>121.3</td>
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**AREA**

**AREA** is the number of square units in a flat region. The formulas to calculate the area of several kinds of polygons are:

- **RECTANGLE**: \( A = bh \)
- **PARALLELOGRAM**: \( A = bh \)
- **TRAPEZOID**: \( A = \frac{1}{2} (b_1 + b_2)h \)
- **TRIANGLE**: \( A = \frac{1}{2} bh \)

Note that the legs of any right triangle form a base and a height for the triangle.

The area of a more complicated figure may be found by breaking it into smaller regions of the types shown above, calculating each area, and finding the sum of the areas.

**Example 1**

Find the area of each figure. All lengths are centimeters.

a) \[
\begin{align*}
A &= bh = (81)(23) = 1863 \text{ cm}^2 
\end{align*}
\]

b) \[
\begin{align*}
A &= \frac{1}{2} (12)(4) = 24 \text{ cm}^2 
\end{align*}
\]

c) \[
\begin{align*}
A &= \frac{1}{2} (108)(42) = 2268 \text{ cm}^2 
\end{align*}
\]

d) \[
\begin{align*}
A &= (21)(8) = 168 \text{ cm}^2 
\end{align*}
\]

Note that 10 is a side of the parallelogram, not the height.
Example 2

Find the area of the shaded region.

The area of the shaded region is the area of the triangle minus the area of the rectangle.

triangle: \[ A = \frac{1}{2}(7)(10) = 35cm^2 \]

rectangle: \[ A = 5(4) = 20cm^2 \]

shaded region: \[ A = 35 - 20 = 15cm^2 \]

Find the area of the following triangles, parallelograms and trapezoids. Pictures are not drawn to scale. Round answers to the nearest tenth.

1. 2. 3. 4.

5. 6. 7. 8.

9. 10. 11. 12.

13. 14. 15. 16.
Find the area of the shaded region.

17. 

18. 

19. 

20. 

Answers (in square units)
1. 100 2. 90 3. 24 4. 15 5. 338 6. 105
7. 93 8. 309.8 9. 126 10. 19.5 11. 36.3 12. 54
You can find where two lines intersect (cross) by using algebraic methods. The two most common methods are **substitution** and **elimination** (also known as the addition method).

**Example 1**

Solve the following system of equations at right by **substitution**.

Check your solution.

When solving a system of equations, you are solving to find the x- and y-values that result in true statements when you substitute them into both equations. You generally find each value one at a time. Since both equations are in y-form (that is, solved for y), and we know $y = y$, we can substitute the right side of each equation for $y$ in the simple equation $y = y$ and write

$$5x + 1 = -3x - 15.$$

Now solve for $x$. $5x + 1 = -3x - 15 \Rightarrow 8x + 1 = -15 \Rightarrow 8x = -16 \Rightarrow x = -2$

Remember you must find $x$ and $y$. To find $y$, use either one of the two original equations. Substitute the value of $x$ into the equation and find the value of $y$. Using the first equation,

$$y = 5x + 1 \Rightarrow y = 5(-2) + 1 \Rightarrow y = -10 + 1 = -9.$$

The solution appears to be $(-2, -9)$. In order for this to be a solution, it must make both equations true when you replace $x$ with -2 and $y$ with -9. Substitute the values in both equations to check.

$$y = 5x + 1$$

$$-9 \neq 5(-2) + 1$$

$$-9 \neq -10 + 1$$

$$-9 = -9 \text{ Check!}$$

Therefore, $(-2, -9)$ is the solution.

**Example 2**
**Substitution** can also be used when the equations are not in y-form.

Use substitution to rewrite the two equations as

\[ 4(-3y + 1) - 3y = -11 \]

by replacing \( x \) with \((-3y + 1)\), then solve for \( y \) as shown at right.

Substitute \( y = 1 \) into \( x = -3y + 1 \). Solve for \( x \), and write the answer for \( x \) and \( y \) as an ordered pair, \((1, -2)\). Substitute \( y = 1 \) into \( 4x - 3y = -11 \) to verify that either original equation may be used to find the second coordinate. Check your answer as shown in example 1.
**Example 3**

When you have a pair of two-variable equations, sometimes it is easier to **eliminate** one of the variables to obtain one single variable equation. You can do this by adding the two equations together as shown in the example below.

Solve the system at right:

\[
\begin{align*}
2x + y &= 11 \\
x - y &= 4
\end{align*}
\]

To eliminate the \(y\) terms, **add** the two equations together.

\[
\begin{align*}
3x &= 15 \\
x &= 5
\end{align*}
\]

then solve for \(x\).

Once we know the \(x\)-value we can substitute it into **either** of the original equations to find the corresponding value of \(y\).

Using the first equation:

\[
\begin{align*}
2x + y &= 11 \\
2(5) + y &= 11 \\
10 + y &= 11 \\
y &= 1
\end{align*}
\]

Check the solution by substituting both the \(x\)-value and \(y\)-value into the other original equation, \(x - y = 4\): \(5 - 1 = 4\), checks.

**Example 4**

You can solve the system of equations at right by elimination, but before you can eliminate one of the variables, you must adjust the coefficients of one of the variables so that they are additive opposites.

To eliminate \(y\), multiply the first equation by 3, then multiply the second equation by \(-2\) to get the equations at right.

\[
\begin{align*}
3x + 2y &= 11 \\
4x + 3y &= 14
\end{align*}
\]

Next eliminate the \(y\) terms by adding the two adjusted equations.

\[
\begin{align*}
9x + 6y &= 33 \\
-8x - 6y &= -28 \\
x &= 5
\end{align*}
\]

Since \(x = 5\), substitute in either original equation to find that \(y = -2\). Therefore, the solution to the system of equations is \((5, -2)\).

You could also solve the system by first multiplying the first equation by 4 and the second equation by \(-3\) to eliminate \(x\), then proceeding as shown above to find \(y\).
Solve the following systems of equations to find the point of intersection \((x, y)\) for each pair of lines.

1. \begin{align*}
y &= x - 6 \\
y &= 12 - x
\end{align*}

2. \begin{align*}
y &= 3x - 5 \\
y &= x + 3
\end{align*}

3. \begin{align*}
x &= 7 + 3y \\
x &= 4y + 5
\end{align*}

4. \begin{align*}
x &= -3y + 10 \\
x &= -6y - 2
\end{align*}

5. \begin{align*}
y &= x + 7 \\
y &= 4x - 5
\end{align*}

6. \begin{align*}
y &= 7 - 3x \\
y &= 2x - 8
\end{align*}

7. \begin{align*}
y &= 3x - 1 \\
2x - 3y &= 10
\end{align*}

8. \begin{align*}
x &= -\frac{1}{2}y + 4 \\
8x + 3y &= 31
\end{align*}

9. \begin{align*}
y &= 4x + 10 \\
6x + 2y &= 10
\end{align*}

10. \begin{align*}
y &= \frac{3}{5}x - 2 \\
y &= \frac{x}{10} + 1
\end{align*}

11. \begin{align*}
y &= -4x + 5 \\
y &= x
\end{align*}

12. \begin{align*}
x &= 4x - 3y = -10 \\
x &= \frac{1}{4}y - 1
\end{align*}

13. \begin{align*}
x + y &= 12 \\
x - y &= 4
\end{align*}

14. \begin{align*}
2x - y &= 6 \\
4x - y &= 12
\end{align*}

15. \begin{align*}
x + 2y &= 7 \\
5x - 4y &= 14
\end{align*}

16. \begin{align*}
5x - 2y &= 6 \\
4x + y &= 10
\end{align*}

17. \begin{align*}
x + y &= 10 \\
x - 2y &= 5
\end{align*}

18. \begin{align*}
3y - 2x &= 16 \\
y &= 2x + 4
\end{align*}

19. \begin{align*}
x + y &= 11 \\
x = y - 3
\end{align*}

20. \begin{align*}
x + 2y &= 15 \\
y &= x - 3
\end{align*}

21. \begin{align*}
y + 5x &= 10 \\
y - 3x &= 14
\end{align*}

22. \begin{align*}
y &= 7x - 3 \\
4x + 2y &= 8
\end{align*}

23. \begin{align*}
y &= 12 - x \\
y &= x - 4
\end{align*}

24. \begin{align*}
y &= 6 - 2x \\
y &= 4x - 12
\end{align*}

### Answers

1. \((9, 3)\)  
2. \((4, 7)\)  
3. \((13, 2)\)  
4. \((22, -4)\)  
5. \((4, 11)\)  
6. \((3, -2)\)  
7. \((-1, -4)\)  
8. \(\left(\frac{3}{2}, 1\right)\)  
9. \((0, 5)\)  
10. \((6, 1.6)\)  
11. \((1, 1)\)  
12. \((-0.25, 3)\)  
13. \((8, 4)\)  
14. \((3, 0)\)  
15. \((4, 1.5)\)  
16. \((2, 2)\)  
17. \(\left(\frac{25}{3}, \frac{5}{3}\right)\)  
18. \((1, 6)\)  
19. \((4, 7)\)  
20. \((7, 4)\)
<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>21.</td>
<td>(-0.5, 12.5)</td>
<td>22.</td>
<td>( \left( \frac{7}{5}, \frac{22}{9} \right) )</td>
<td>23.</td>
</tr>
</tbody>
</table>
PROPERTIES OF ANGLES, LINES, AND TRIANGLES

Parallel lines

1

2

3

4

Triangles

6

7

8

9

10

11

• corresponding angles are equal:
  \( m\angle 1 = m\angle 3 \)

• alternate interior angles are equal:
  \( m\angle 2 = m\angle 3 \)

• \( m\angle 2 + m\angle 4 = 180^\circ \)

• \( m\angle 7 + m\angle 8 + m\angle 9 = 180^\circ \)

• \( m\angle 6 = m\angle 8 + m\angle 9 \)
  (exterior angle = sum remote interior angles)

• \( m\angle 10 + m\angle 11 = 90^\circ \)
  (complementary angles)

Also shown in the above figures:

• vertical angles are equal: \( m\angle 1 = m\angle 2 \)

• linear pairs are supplementary:
  \( m\angle 3 + m\angle 4 = 180^\circ \)
  and \( m\angle 6 + m\angle 7 = 180^\circ \)

In addition, an isosceles triangle, \( \triangle ABC \), has
\( BA = BC \) and \( m\angle A = m\angle C \). An equilateral triangle, \( \triangle GFH \), has \( GF = FH = HG \) and
\( m\angle G = m\angle F = m\angle H = 60^\circ \).

Example 1

Solve for \( x \).

Use the Exterior Angle Theorem: \( 6x + 8^\circ = 49^\circ + 67^\circ \)

\( 6x = 108^\circ \Rightarrow x = \frac{108^\circ}{6} \Rightarrow x = 18^\circ \)

Example 2

Solve for \( x \).

There are a number of relationships in this diagram. First, \( \angle 1 \) and the \( 127^\circ \) angle are supplementary, so we know that \( m\angle 1 + 127^\circ = 180^\circ \) so \( m\angle 1 = 53^\circ \). Using the same idea, \( m\angle 2 = 47^\circ \). Next, \( m\angle 3 + 53^\circ + 47^\circ = 180^\circ \), so \( m\angle 3 = 80^\circ \). Because angle 3 forms a vertical pair with the angle marked \( 7x + 3^\circ \), \( 80^\circ = 7x + 3^\circ \), so \( x = 11^\circ \).
**Example 3**

Find the measure of the acute alternate interior angles.

Parallel lines mean that alternate interior angles are equal, so $5x + 28 = 2x + 46 \Rightarrow 3x = 18 \Rightarrow x = 6$. Use either algebraic angle measure: $2(6) + 46 = 58$ for the measure of the acute angle.
Use the geometric properties and theorems you have learned to solve for x in each diagram and write the property or theorem you use in each case.

1. \[ \triangle \quad 75 \degree \quad 60 \degree \quad x \degree \]
2. \[ \triangle \quad 65 \degree \quad 80 \degree \quad x \degree \]
3. \[ \triangle \quad 100 \degree \quad x \degree \quad x \degree \]
4. \[ \triangle \quad 112 \degree \quad x \degree \quad x \degree \]
5. \[ \triangle \quad 60 \degree \quad 4x + 10 \degree \quad x \degree \]
6. \[ \triangle \quad 60 \degree \quad 8x - 60 \degree \quad x \degree \]
7. \[ \triangle \quad 45 \degree \quad 3x \degree \]
8. \[ \triangle \quad 125 \degree \quad 5x \degree \]
9. \[ \triangle \quad 68 \degree \quad 5x + 12 \degree \]
10. \[ \triangle \quad 128 \degree \quad 10x + 2 \degree \]
11. \[ \triangle \quad 30 \degree \quad 19x + 3 \degree \]
12. \[ \triangle \quad 52 \degree \quad 128 \degree \]
13. \[ \triangle \quad 142 \degree \quad 38 \degree \]
14. \[ \triangle \quad 142 \degree \quad 38 \degree \]
15. \[ \triangle \quad 128 \degree \quad 52 \degree \]
16. \[ \triangle \quad 5x + 3 \degree \quad 128 \degree \]
17. \[ \triangle \quad x \degree \quad 23 \degree \]
18. \[ \triangle \quad 8x \degree \quad 117 \degree \]
19. \[ \triangle \quad 18 \degree \quad 5x + 36 \degree \]
20. \[ \triangle \quad 8x + 15 \degree \quad 12 \degree \]
21. \[ \triangle \quad 8x - 18 \text{ cm} \quad 5x + 3 \text{ cm} \]
22. 

\[ 5x - 18 \]

23. 

\[ \frac{70}{50} \sqrt{x - 10} \]

24. 

\[ \frac{13x + 20}{15x - 20} \]

25. 

\[ 40 \sqrt{3x - 20} \]

26. 

\[ 45 \sqrt{2x + 5} \]

27. 

\[ 7x - 4 \sqrt{5x + 8} \]

28. 

\[ 7x - 4 \sqrt{6x - 4} \]

\[ 5x + 8 \]

\[ 7x - 4 \sqrt{6x - 4} \]

Answers

1. 45°  2. 35°  3. 40°  4. 34°  5. 12.5°  6. 15°  7. 15°  8. 25°  9. 20°  10. 5°  11. 3°  12. 10 \frac{2}{3}°  13. 7°  14. 2°  15. 7°  16. 25°  17. 81°  18. 7.5°  19. 9°  20. 7.5°  21. 7°  22. 15.6°  23. 26°  24. 2°  25. 40°  26. 65°  27. 7 \frac{1}{6}°  28. 10°
LINEAR INEQUALITIES

To graph a linear inequality, first graph the line of the corresponding equality. This line is known as the dividing line, since all the points that make the inequality true lie on one side or the other of the line. Before you draw the line, decide whether the dividing line is part of the solution or not, that is, whether the line is solid or dashed. If the inequality symbol is either ≤ or ≥, then the dividing line is part of the inequality and it must be solid. If the symbol is either < or >, then the dividing line is not part of the inequality and it must be dashed.

Next, decide which side of the dividing line must be shaded to show the part of the graph that represents all values that make the inequality true. Choose a point not on the dividing line. Substitute this point into the original inequality. If the inequality is true for this test point, then shade the graph on this side of the dividing line. If the inequality is false for the test point, then shade the opposite side of the dividing line.

CAUTION: If the inequality is not in slope-intercept form and you have to solve it for y, always use the original inequality to test a point, NOT the solved form.

Example 1

Graph the inequality \( y > 3x - 2 \).
First, graph the line \( y = 3x - 2 \), but draw it dashed since > means the dividing line is not part of the solution.

Next, test the point (-2, 4) to the left of the dividing line.

\[ 4 > 3(-2) - 2, \text{ so } 4 > -8 \]

Since the inequality is true for this point, shade the left side of the dividing line.
Example 2

Graph the system of inequalities
\[ y \leq \frac{1}{2} x + 2 \quad \text{and} \quad y > -\frac{2}{3} x - 1. \]

Graph the lines \( y = \frac{1}{2} x + 2 \) and \( y = -\frac{2}{3} x - 1 \). The first line is solid, the second is dashed. Test the point \((-4, 5)\) in the first inequality.

\[ 5 \leq \frac{1}{2}(-4) + 2, \quad \text{so} \quad 5 \leq 0 \]

This inequality is false, so shade on the opposite side of the dividing line, from \((-4, 5)\). Test the same point in the second inequality.

\[ 5 > -\frac{2}{3}(-4) - 1, \quad \text{so} \quad 5 > \frac{5}{3} \]

This inequality is true, so shade on the same side of the dividing line as \((-4, 5)\).

The solution is the overlap of the two shaded regions shown by the darkest shading in the second graph above right.

Graph each of the following inequalities on separate sets of axes.

1. \( y \leq 3x + 1 \)
2. \( y \geq 2x - 1 \)
3. \( y \geq -2x - 3 \)
4. \( y \leq -3x + 4 \)
5. \( y > 4x + 2 \)
6. \( y < 2x + 1 \)
7. \( y < -3x - 5 \)
8. \( y > -5x - 4 \)
9. \( y \leq 3 \)
10. \( y \geq -2 \)
11. \( x > 1 \)
12. \( x \leq 8 \)
13. \( y \geq \frac{2}{3} x + 8 \)
14. \( y \leq -\frac{2}{3} x + 3 \)
15. \( y < \frac{3}{5} x - 7 \)
16. \( y \geq \frac{1}{4} x - 2 \)
17. \( 3x + 2y \geq 7 \)
18. \( 2x - 3y \leq 5 \)
19. \(-4x + 2y < 3 \)
20. \(-3x - 4y > 4 \)

Graph each of the following pairs of inequalities on the same set of axes.

21. \( y > 3x - 4 \) and \( y \leq -2x + 5 \)
22. \( y \geq -3x - 6 \) and \( y > 4x - 4 \)
23. \( y \leq -\frac{3}{5} x + 4 \) and \( y \leq \frac{1}{3} x + 3 \)
24. \( y < -\frac{3}{7} x - 1 \) and \( y > \frac{4}{5} x + 1 \)
25. \( y < 3 \) and \( y \leq -\frac{1}{2}x + 2 \)

26. \( x \leq 3 \) and \( y < \frac{3}{4}x - 4 \)

Write an inequality for each of the following graphs.

27. 

28. 

29. 

30. 

31. 

32. 

Answers

1. 

2. 

3. 

4. 

5. 

6.
27. \( y \leq x + 5 \)
28. \( y \geq -\frac{5}{2}x + 5 \)
29. \( y \leq \frac{1}{3}x + 4 \)
30. \( y > -4x + 1 \)
31. \( x \geq -4 \)
32. \( y \leq 3 \)
MULITPLYING POLYNOMIALS

We can use generic rectangles as area models to find the products of polynomials. A generic rectangle helps us organize the problem. It does not have to be drawn accurately or to scale.

Example 1
Multiply \((2x + 5)(x + 3)\)

\[
\begin{array}{c|c|c}
   x & 2x & + 5 \\
+ & + & + \\
3 & 6x & 15 \\
\end{array}
\]

\((2x + 5)(x + 3) = 2x^2 + 11x + 15\)

Example 2
Multiply \((x + 9)(x^2 - 3x + 5)\)

\[
\begin{array}{c|c|c|c|c}
   x & x^3 & -3x^2 & 5x \\
+ & + & + & + \\
9 & 9x^2 & -27x & 45 \\
\end{array}
\]

Therefore \((x + 9)(x^2 - 3x + 5) = x^3 + 9x^2 - 3x^2 - 27x + 5x + 45 = x^3 + 6x^2 - 22x + 45\)

Another approach to multiplying binomials is to use the mnemonic "F.O.I.L." F.O.I.L. is an acronym for First, Outside, Inside, Last in reference to the positions of the terms in the two binomials.

Example 3
Multiply \((3x - 2)(4x + 5)\) using the F.O.I.L. method.

- **F.** multiply the FIRST terms of each binomial \((3x)(4x) = 12x^2\)
- **O.** multiply the OUTSIDE terms \((3x)(5) = 15x\)
- **I.** multiply the INSIDE terms \((-2)(4x) = -8x\)
- **L.** multiply the LAST terms of each binomial \((-2)(5) = -10\)

Finally, we combine like terms: \(12x^2 + 15x - 8x - 10 = 12x^2 + 7x - 10\).
Find each of the following products.

1. \((3x + 2)(2x + 7)\)  
2. \((4x + 5)(5x + 3)\)  
3. \((2x - 1)(3x + 1)\)  
4. \((2a - 1)(4a + 7)\)  
5. \((m - 5)(m + 5)\)  
6. \((y - 4)(y + 4)\)  
7. \((3x - 1)(x + 2)\)  
8. \((3a - 2)(a - 1)\)  
9. \((2y - 5)(y + 4)\)  
10. \((3t - 1)(3t + 1)\)  
11. \((3y - 5)^2\)  
12. \((4x - 1)^2\)  
13. \((2x + 3)^2\)  
14. \((5n + 1)^2\)  
15. \((3x - 1)(2x^2 + 4x + 3)\)  
16. \((2x + 7)(4x^2 - 3x + 2)\)  
17. \((x + 7)(3x^2 - x + 5)\)  
18. \((x - 5)(x^2 - 7x + 1)\)  
19. \((3x + 2)(x^3 - 7x^2 + 3x)\)  
20. \((2x + 3)(3x^2 + 2x - 5)\)

Answers

1. \(6x^2 + 25x + 14\)  
2. \(20x^2 + 37x + 15\)  
3. \(6x^2 - x - 1\)  
4. \(8a^2 + 10a - 7\)  
5. \(m^2 - 25\)  
6. \(y^2 - 16\)  
7. \(3x^2 + 5x - 2\)  
8. \(3a^2 - 5a + 2\)  
9. \(2y^2 + 3y - 20\)  
10. \(9t^2 - 1\)  
11. \(9y^2 - 30y + 25\)  
12. \(16x^2 - 8x + 1\)  
13. \(4x^2 + 12x + 9\)  
14. \(25n^2 + 10n + 1\)  
15. \(6x^3 + 10x^2 + 5x - 3\)  
16. \(8x^3 + 22x^2 - 17x + 14\)  
17. \(3x^3 + 20x^2 - 2x + 35\)  
18. \(x^3 - 12x^2 + 36x - 5\)  
19. \(3x^4 - 19x^3 - 5x^2 + 6x\)  
20. \(6x^3 + 13x^2 - 4x - 15\)
Often we want to un-multiply or **factor** a polynomial $P(x)$. This process involves finding a constant and/or another polynomial that evenly divides the given polynomial. In formal mathematical terms, this means $P(x) = q(x) \cdot r(x)$, where $q$ and $r$ are also polynomials. For elementary algebra there are three general types of factoring.

1) **Common term** (finding the largest common factor):
   
   $6x + 18 = 6(x + 3)$ where 6 is a common factor of both terms.
   
   $2x^3 - 8x^2 - 10x = 2x(x^2 - 4x - 5)$ where 2x is the common factor.
   
   $2x^2(x - 1) + 7(x - 1) = (x - 1)(2x^2 + 7)$ where $x - 1$ is the common factor.

2) **Special products**
   
   $a^2 - b^2 = (a + b)(a - b)$
   
   $x^2 - 25 = (x + 5)(x - 5)$
   
   $6x^2 - 4y^2 = (3x + 2y)(3x - 2y)$
   
   $x^2 + 2xy + y^2 = (x + y)^2$
   
   $x^2 + 8x + 16 = (x + 4)^2$
   
   $x^2 - 2xy + y^2 = (x - y)^2$
   
   $x^2 - 8x + 16 = (x - 4)^2$

3a) **Trinomials** in the form $x^2 + bx + c$ where the coefficient of $x^2$ is 1.

   Consider $x^2 + (d + e)x + d \cdot e = (x + d)(x + e)$, where the coefficient of $x$ is the **sum** of two numbers $d$ and $e$ AND the constant is the **product** of the same two numbers, $d$ and $e$. A quick way to determine all of the possible pairs of integers $d$ and $e$ is to factor the constant in the original trinomial. For example, 12 is $1 \cdot 12$, $2 \cdot 6$, and $3 \cdot 4$. The signs of the two numbers are determined by the combination you need to get the sum. The "sum and product" approach to factoring trinomials is the same as solving a "Diamond Problem" in CPM's Algebra 1 course (see below).

   $x^2 + 8x + 15 = (x + 3)(x + 5)$; $3 + 5 = 8$, $3 \cdot 5 = 15$
   
   $x^2 - 2x - 15 = (x - 5)(x + 3)$; $-5 + 3 = -2$, $-5 \cdot 3 = -15$
   
   $x^2 - 7x + 12 = (x - 3)(x - 4)$; $-3 + (-4) = -7$, $(-3)(-4) = 12$

   The sum and product approach can be shown visually using rectangles for an area model. The figure at far left below shows the "Diamond Problem" format for finding a sum and product. Here is how to use this method to factor $x^2 + 6x + 8$.

>> Explanation and examples continue on the next page. >>
3b) **Trinomials** in the form $ax^2 + bx + c$ where $a \neq 1$.

Note that the upper value in the diamond is no longer the constant. Rather, it is the product of $a$ and $c$, that is, the coefficient of $x^2$ and the constant.

Below is the process to factor $5x^2 - 13x + 6$.

Polynomials with four or more terms are generally factored by grouping the terms and using one or more of the three procedures shown above. Note that polynomials are usually factored completely. In the second example in part (1) above, the trinomial also needs to be factored. Thus, the complete factorization of $2x^3 - 8x^2 - 10x = 2x(x^2 - 4x - 5) = 2x(x - 5)(x + 1)$.

Factor each polynomial completely.

1. $x^2 - x - 42$  
2. $4x^2 - 18$  
3. $2x^2 + 9x + 9$  
4. $2x^2 + 3xy + y^2$  
5. $6x^2 - x - 15$  
6. $4x^2 - 25$  
7. $x^2 - 28x + 196$  
8. $7x^2 - 847$  
9. $x^2 + 18x + 81$  
10. $x^2 + 4x - 21$  
11. $3x^2 + 21x$  
12. $3x^2 - 20x - 32$  
13. $9x^2 - 16$  
14. $4x^2 + 20x + 25$  
15. $x^2 - 5x + 6$  
16. $5x^3 + 15x^2 - 20x$  
17. $4x^2 + 18$  
18. $x^2 - 12x + 36$  
19. $x^2 - 3x - 54$  
20. $6x^2 - 21$  
21. $2x^2 + 15x + 18$  
22. $16x^2 - 1$  
23. $x^2 - 14x + 49$  
24. $x^2 + 8x + 15$  
25. $3x^3 - 12x^2 - 45x$  
26. $3x^2 + 24$  
27. $x^2 + 16x + 64$  

Factor completely.

28. $75x^3 - 27x$  
29. $3x^3 - 12x^2 - 36x$  
30. $4x^3 - 44x^2 + 112x$  
31. $5y^2 - 125$  
32. $3x^2y^2 - xy^2 - 4y^2$  
33. $x^3 + 10x^2 - 24x$  
34. $3x^3 - 6x^2 - 45x$  
35. $3x^2 - 27$  
36. $x^4 - 16$
Factor each of the following completely. Use the modified diamond approach.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>37.</td>
<td>$2x^2 + 5x - 7$</td>
<td>38.</td>
</tr>
<tr>
<td>40.</td>
<td>$4x^2 - 13x + 3$</td>
<td>41.</td>
</tr>
<tr>
<td>43.</td>
<td>$64x^2 + 16x + 1$</td>
<td>44.</td>
</tr>
</tbody>
</table>

**Answers**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$(x + 6)(x - 7)$</td>
<td>2.</td>
</tr>
<tr>
<td>4.</td>
<td>$(2x + y)(x + y)$</td>
<td>5.</td>
</tr>
<tr>
<td>7.</td>
<td>$(x - 14)^2$</td>
<td>8.</td>
</tr>
<tr>
<td>10.</td>
<td>$(x + 7)(x - 3)$</td>
<td>11.</td>
</tr>
<tr>
<td>13.</td>
<td>$(3x - 4)(3x + 4)$</td>
<td>14.</td>
</tr>
<tr>
<td>16.</td>
<td>$5x(x + 4)(x - 1)$</td>
<td>17.</td>
</tr>
<tr>
<td>19.</td>
<td>$(x - 9)(x + 6)$</td>
<td>20.</td>
</tr>
<tr>
<td>22.</td>
<td>$(4x + 1)(4x - 1)$</td>
<td>23.</td>
</tr>
<tr>
<td>25.</td>
<td>$3x(x^2 - 4x - 15)$</td>
<td>26.</td>
</tr>
<tr>
<td>28.</td>
<td>$3x(5x - 3)(5x + 3)$</td>
<td>29.</td>
</tr>
<tr>
<td>31.</td>
<td>$5(y + 5)(y - 5)$</td>
<td>32.</td>
</tr>
<tr>
<td>34.</td>
<td>$3x(x - 5)(x + 3)$</td>
<td>35.</td>
</tr>
<tr>
<td>37.</td>
<td>$(2x + 7)(x - 1)$</td>
<td>38.</td>
</tr>
<tr>
<td>40.</td>
<td>$(4x - 1)(x - 3)$</td>
<td>41.</td>
</tr>
<tr>
<td>43.</td>
<td>$(8x + 1)^2$</td>
<td>44.</td>
</tr>
</tbody>
</table>
ZERO PRODUCT PROPERTY AND QUADRATICS

If \( a \cdot b = 0 \), then either \( a = 0 \) or \( b = 0 \).

Note that this property states that at least one of the factors MUST be zero. It is also possible that all of the factors are zero. This simple statement gives us a powerful result which is most often used with equations involving the products of binomials. For example, solve \( (x + 5)(x - 2) = 0 \).

By the Zero Product Property, since \( (x + 5)(x - 2) = 0 \), either \( x + 5 = 0 \) or \( x - 2 = 0 \). Thus, \( x = -5 \) or \( x = 2 \).

The Zero Product Property can be used to find where a quadratic crosses the x-axis. These points are the x-intercepts. In the example above, they would be \((-5, 0)\) and \((2, 0)\).

**Example 1**

Where does \( y = (x + 3)(x - 7) \) cross the x-axis? Since \( y = 0 \) at the x-axis, then \( (x + 3)(x - 7) = 0 \) and the Zero Product Property tells you that \( x = -3 \) and \( x = 7 \) so \( y = (x + 3)(x - 7) \) crosses the x-axis at \((-3, 0)\) and \((7, 0)\).

**Example 2**

Where does \( y = x^2 - x - 6 \) cross the x-axis? First factor \( x^2 - x - 6 \) into \( (x + 2)(x - 3) \) to get \( y = (x + 2)(x - 3) \). By the Zero Product Property, the x-intercepts are \((-2, 0)\) and \((3, 0)\).
Example 3
Graph \( y = x^2 - x - 6 \)

Since you know the \( x \)-intercepts from example 2, you already have two points to graph. You need a table of values to get additional points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>-4</td>
<td>-6</td>
<td>-6</td>
<td>-4</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

Example 4
Graph \( y > x^2 - x - 6 \)

First graph \( y = x^2 - x - 6 \) as you did at left. Use a dashed curve. Second, pick a point not on the parabola and substitute it into the inequality. For example, testing point \((0, 0)\) in \( y > x^2 - x - 6 \) gives \( 0 > -6 \) which is a true statement. This means that \((0, 0)\) is a solution to the inequality as well as all points inside the curve. Shade the interior of the parabola.
Solve the following problems using the Zero Product Property.

1. \((x - 2)(x + 3) = 0\)
2. \(2(x + 5)(x + 6) = 0\)
3. \((x - 18)(x - 3) = 0\)
4. \(4x^2 - 5x - 6 = 0\)
5. \((2x - 1)(x + 2) = 0\)
6. \(2x(x - 3)(x + 4) = 0\)
7. \(3x^2 - 13x - 10 = 0\)
8. \(2x^2 - x = 15\)

Use factoring and the Zero Product Property to find the x-intercepts of each parabola below. Express your answer as ordered pair(s).

9. \(y = x^2 - 3x + 2\)
10. \(y = x^2 - 10x + 25\)
11. \(y = x^2 - x - 12\)
12. \(y = x^2 - 4x - 5\)
13. \(y = x^2 + 2x - 8\)
14. \(y = x^2 + 6x + 9\)
15. \(y = x^2 - 8x + 16\)
16. \(y = x^2 - 9\)

Graph the following inequalities. Be sure to use a test point to determine which region to shade. Your solutions to the previous problems might be helpful.

17. \(y < x^2 - 3x + 2\)
18. \(y > x^2 - 10x + 25\)
19. \(y \leq x^2 - x - 12\)
20. \(y \geq x^2 - 4x - 5\)
21. \(y > x^2 + 2x - 8\)
22. \(y \geq x^2 + 6x + 9\)
23. \(y < x^2 - 8x + 16\)
24. \(y \leq x^2 - 9\)

Answers

1. \(x = 2, x = -3\)
2. \(x = 0, x = -5, x = -6\)
3. \(x = 18, x = 3\)
4. \(x = -0.75, x = 2\)
5. \(x = 0.5, x = -2\)
6. \(x = 0, x = 3, x = -4\)
7. \(x = -\frac{2}{3}, x = 5\)
8. \(x = -2.5, x = 3\)
9. \((1, 0)\) and \((2, 0)\)
10. \((5, 0)\)
11. \((-3, 0)\) and \((4, 0)\)
12. \((5, 0)\) and \((-1, 0)\)
13. \((-4, 0)\) and \((2, 0)\)
14. \((-3, 0)\)
15. \((4, 0)\)
16. \((3, 0)\) and \((-3, 0)\)
THE QUADRATIC FORMULA

You have used factoring and the Zero Product Property to solve quadratic equations. You can solve any quadratic equation by using the QUADRATIC FORMULA.

If \( ax^2 + bx + c = 0 \), then \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

For example, suppose \( 3x^2 + 7x - 6 = 0 \). Here \( a = 3 \), \( b = 7 \), and \( c = -6 \).
Substituting these values into the formula results in:

\[
x = \frac{-7 \pm \sqrt{7^2 - 4(3)(-6)}}{2(3)} \Rightarrow x = \frac{-7 \pm \sqrt{21}}{6} \Rightarrow x = \frac{-7 \pm 11}{6}
\]

Remember that non-negative numbers have both a positive and negative square root.
The sign \( \pm \) represents this fact for the square root in the formula and allows us to write the equation once (representing two possible solutions) until later in the solution process.

Split the numerator into the two values:

\[
x = \frac{-7 + 11}{6} \quad \text{or} \quad x = \frac{-7 - 11}{6}
\]

Thus the solution for the quadratic equation is: \( x = \frac{2}{3} \) or -3.

Example 1

Solve \( x^2 + 3x - 2 = 0 \) using the quadratic formula.

First, identify the values for \( a \), \( b \), and \( c \). In this case they are 1, 3, and -2, respectively. Next, substitute these values into the quadratic formula.

\[
x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)} \Rightarrow x = \frac{-3 \pm \sqrt{17}}{2}
\]

Then split the numerator into the two values:

\[
x = \frac{-3 + \sqrt{17}}{2} \quad \text{or} \quad x = \frac{-3 - \sqrt{17}}{2}
\]

Using a calculator, the solution for the quadratic equation is: \( x = 0.56 \) or -3.56.

Example 2

Solve \( 4x^2 + 4x = 3 \) using the quadratic formula.

To solve any quadratic equation it must first be equal to zero. Rewrite the equation as \( 4x^2 + 4x - 3 = 0 \). Identify the values for \( a \), \( b \), and \( c \): 4, 4, and -3, respectively. Substitute these values into the quadratic formula.

\[
x = \frac{-4 \pm \sqrt{4^2 - 4(4)(-3)}}{2(4)} \Rightarrow x = \frac{-4 \pm \sqrt{64}}{8} \Rightarrow x = \frac{-4 \pm 8}{8}
\]
Split the numerator into the two values: $x = \frac{-4 + 8}{8}$ or $x = \frac{-4 - 8}{8}$, so $x = \frac{1}{2}$ or $-\frac{3}{2}$. 
Use the quadratic formula to solve each of the following equations.

1. \( x^2 - x - 6 = 0 \)
2. \( x^2 + 8x + 15 = 0 \)
3. \( x^2 + 13x + 42 = 0 \)
4. \( x^2 - 10x + 16 = 0 \)
5. \( x^2 + 5x + 4 = 0 \)
6. \( x^2 - 9x + 18 = 0 \)
7. \( 5x^2 - x - 4 = 0 \)
8. \( 4x^2 - 11x - 3 = 0 \)
9. \( 6x^2 - x - 15 = 0 \)
10. \( 6x^2 + 19x + 15 = 0 \)
11. \( 3x^2 + 5x - 28 = 0 \)
12. \( 2x^2 - x - 14 = 0 \)
13. \( 4x^2 - 9x + 4 = 0 \)
14. \( 2x^2 - 5x + 2 = 0 \)
15. \( 20x^2 + 20x = 1 \)
16. \( 13x^2 - 16x = 4 \)
17. \( 7x^2 + 28x = 0 \)
18. \( 5x^2 = -125x \)
19. \( 8x^2 - 50 = 0 \)
20. \( 15x^2 = 3 \)

**Answers**

1. \( x = -2, 3 \)
2. \( x = -5, -3 \)
3. \( x = -7, -6 \)
4. \( x = 2, 8 \)
5. \( x = -4, -1 \)
6. \( x = 3, 6 \)
7. \( x = -\frac{4}{5}, 1 \)
8. \( x = -\frac{1}{4}, 3 \)
9. \( x = -\frac{3}{2}, -\frac{5}{3} \)
10. \( x = -\frac{3}{2}, -\frac{5}{3} \)
11. \( x = -4, \frac{7}{3} \)
12. \( x = \frac{1 \pm \sqrt{113}}{4} \)
13. \( x = \frac{9 \pm \sqrt{17}}{8} \)
14. \( x = 2, \frac{1}{2} \)
15. \( x = -\frac{20 \pm \sqrt{480}}{40} = -\frac{5 \pm \sqrt{10}}{10} \)
16. \( x = \frac{16 \pm \sqrt{464}}{26} = \frac{8 \pm 2\sqrt{29}}{13} \)
17. \( x = -4, 0 \)
18. \( x = -25, 0 \)
19. \( x = \frac{5}{2}, \frac{5}{2} \)
20. \( x = \pm \frac{\sqrt{5}}{5} \)
If two triangles are congruent, then all six of their corresponding pairs of parts are also congruent. In other words, if $\triangle ABC \cong \triangle XYZ$, then all the congruence statements at right are correct. Note that the matching parts follow the order of the letters as written in the triangle congruence statement. For example, if $A$ is first in one statement and $X$ is first in the other, then $A$ always corresponds to $X$ in congruence statements for that relationship or figure.

You only need to know that three pairs of parts are congruent (sometimes in a certain order) to prove that the two triangles are congruent. The Triangle Congruence Properties are: SSS (Side-Side-Side), SAS (Side-Angle-Side; must be in this order), ASA (Angle-Side-Angle), and HL (Hypotenuse-Leg). AAS is an acceptable variation of ASA.

The four Triangle Congruence Properties are the only correspondences that may be used to prove that two triangles are congruent. Once two triangles are know to be congruent, then the other pairs of their corresponding parts are congruent. In this course, you may justify this conclusion with the statement "congruent triangles give us congruent parts." Remember: only use this statement after you have shown the two triangles are congruent.

**Example 1**
In the figure at left, $CA \cong DO$, $CT \cong DG$, and $AT \cong OG$. Thus $\triangle CAT \cong \triangle DOG$ because of SSS. Now that the triangles are congruent, it is also true that $\angle C \cong \angle D$, $\angle A \cong \angle O$, and $\angle T \cong \angle G$.

**Example 2**
In the figure at right, $RD \cong CP$, $\angle D \cong \angle P$, and $ED \cong AP$. Thus $\triangle RED \cong \triangle CAP$ because of SAS. Now $RE \cong CA$, $\angle R \cong \angle C$, and $\angle E \cong \angle A$.

**Example 3**
In the figure at left, $\angle O \cong \angle A$, $\overline{OX} \cong \overline{AR}$, and $\angle X \cong \angle R$. Thus $\triangle BOX \cong \triangle CAR$ because of ASA. Now $\angle B \cong \angle C$, $\overline{BO} \cong \overline{CA}$, and $\overline{BX} \cong \overline{CR}$.
Example 4

In the figure at right, there are two right angles \( m\angle N = m\angle Y = 90^\circ \) and \( \overline{MO} \cong \overline{KE} \), and \( \overline{ON} \cong \overline{EY} \). Thus, \( \triangle MON \cong \triangle KEY \) because of HL, so \( \overline{MN} \cong \overline{KY} \), \( \angle M \cong \angle K \), and \( \angle O \cong \angle E \).

Example 5

Refer to page 196 in the textbook (problem CG-41) and do each part, (a) through (i). Brief explanations and the answers appear below.

a) \( \triangle PQR \cong \triangle ZXY \) by SSS

b) \( \angle BAE \cong \angle DEC \) (vertical angles), \( \triangle BAE \cong \triangle DEC \) by SAS.

d) These triangles are not necessarily congruent because we don’t know the relative sizes of the sides. Remember that AAA is not a congruence property.

e) \( \angle E \cong \angle H \) because the three angles in any triangle must add up to 180°. \( \triangle DEF \cong \triangle GHJ \) by SAS.

f) \( \triangle YXW \cong \triangle LKJ \) by SAS.

g) Not enough information; pairs of corresponding parts.

h) \( \triangle QSR \cong \triangle PQS \) by ASA.

i) Not necessarily congruent since the one side we know is not between the two angles of the second triangle.

Briefly explain if each of the following pairs of triangles are congruent or not. If so, state the Triangle Congruence Property that supports your conclusion.

1. 

2. 

3.
Answers

1. \( \triangle ABC \cong \triangle DEF \) by ASA
2. \( \triangle GIH \cong \triangle LJK \) by SAS
3. \( \triangle PNM \cong \triangle PNO \) by SSS
4. \( \overline{QS} \cong \overline{QS} \), so \( \triangle QRS \cong \triangle QTS \) by HL
5. The triangles are not necessarily congruent.
6. \( \triangle ABC \cong \triangle DFE \) by ASA or AAS.
7. \( \overline{GI} \cong \overline{GI} \), so \( \triangle GHI \cong \triangle IJG \) by SSS.
8. Alternate interior angles = used twice, so \( \triangle KLN \cong \triangle NMK \) by ASA.
9. Vertical angles = at 0, so \( \triangle POQ \cong \triangle ROS \) by SAS.
10. Vertical angles and/or alternate interior angles =, so \( \triangle TUX \cong \triangle VWX \) by ASA.

11. No, the length of each hypotenuse is different.

12. Pythagorean Theorem, so \( \triangle EGH \cong \triangle IHG \) by SSS.

13. Sum of angles of triangle = 180°, but since the equal angles do not correspond, the triangles are not congruent.

14. \( AF + FC = FC + CD \), so \( \triangle ABC \cong \triangle DEF \) by SSS.

15. \( \overline{XZ} \cong \overline{XZ} \), so \( \triangle WXZ \cong \triangle YXZ \) by AAS.
LAWS OF EXPONENTS

BASE, EXponent, AND VALUE

In the expression $2^5$, 2 is the base, 5 is the exponent, and the value is 32.

$2^5$ means $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

$x^3$ means $x \cdot x \cdot x$

LAWS OF EXPONENTS

Here are the basic patterns with examples:

1) $x^a \cdot x^b = x^{a+b}$
   examples: $x^3 \cdot x^4 = x^{3+4} = x^7$; $2^7 \cdot 2^4 = 2^{11}$

2) $\frac{x^a}{x^b} = x^{a-b}$
   examples: $x^{10} \div x^4 = x^{10-4} = x^6$; $\frac{2^4}{2^7} = 2^{-3}$

3) $(x^a)^b = x^{ab}$
   examples: $(x^4)^3 = x^{4 \cdot 3} = x^{12}$; $(2x^3)^4 = 2^4 \cdot x^{12} = 16x^{12}$

4) $x^{-a} = \frac{1}{x^a}$ and $\frac{1}{x^{-b}} = x^b$
   examples: $3x^{-3}y^2 = \frac{3y^2}{x^3}$; $\frac{2x^5}{y^2} = 2x^5y^2$

Example 1

Simplify: $(2xy^3)(5x^2y^4)$

Multiply the coefficients: $2 \cdot 5 \cdot xy^3 \cdot x^2y^4 = 10xy^{3+4}$

Add the exponents of $x$, then $y$: $10x^{1+2}y^{3+4} = 10x^3y^7$

Example 2

Simplify: $\frac{14x^2y^{12}}{7x^3y^7}$

Divide the coefficients: $\frac{(14+7)x^2y^{12}}{x^3y^7} = \frac{2x^2y^{12}}{x^3y^7}$

Subtract the exponents: $2x^{2-3}y^{12-7} = 2x^{-1}y^5$ OR $\frac{2y^5}{x}$

Example 3

Simplify: $(3x^2y^4)^3$

Cube each factor: $3^3 \cdot (x^2)^3 \cdot (y^4)^3 = 27(x^6)(y^{12})$

Multiply the exponents: $27x^6y^{12}$
Simplify each expression:

1. \( y^5 \cdot y^7 \)  
2. \( b^4 \cdot b^3 \cdot b^2 \)  
3. \( 8^6 \cdot 8^2 \)  
4. \( (y^5)^2 \)

5. \( (3a)^4 \)  
6. \( \frac{m^8}{m^3} \)  
7. \( \frac{12x^9}{4x^4} \)  
8. \( (x^3y^2)^3 \)

9. \( \frac{(v^4)^2}{\sqrt[2]{v^2}} \)  
10. \( \frac{15x^2y^7}{3x^4v^5} \)  
11. \( (4e^4)(ae^{-3})(3a^5c) \)  
12. \( (7x^3y^5)^2 \)

13. \( (4xy^2)(2y)^3 \)  
14. \( \left( \frac{4}{x^2} \right)^3 \)  
15. \( \frac{2a^7(3a^2)}{6a^5} \)  
16. \( \left( \frac{5m^3n}{m^5} \right)^3 \)

17. \( (3a^2x^3)^2(2ax^4)^3 \)  
18. \( \left( \frac{x^3}{y^4} \right)^4 \)  
19. \( \left( \frac{6y^2x^8}{12x^3y^7} \right)^2 \)  
20. \( \frac{(2x^3y^3)(4xy^4)^2}{8x^7y^{12}} \)

Answers

1. \( y^{12} \)  
2. \( b^9 \)  
3. \( 8^8 \)  
4. \( y^{10} \)

5. \( 8la^4 \)  
6. \( m^5 \)  
7. \( 3x^5 \)  
8. \( x^9y^6 \)

9. \( y^2 \)  
10. \( \frac{5y^2}{x^2} \)  
11. \( 12a^6c^8 \)  
12. \( 49x^6y^{10} \)

13. \( 32xy^5 \)  
14. \( \frac{64}{v^6} \)  
15. \( a^6 \)  
16. \( \frac{125n^3}{m^6} \)

17. \( 72a^7x^{18} \)  
18. \( \frac{x^{12}}{\sqrt[12]{y^2}} \)  
19. \( \frac{x^{10}}{4u^{10}} \)  
20. \( 16x^{10}y^5 \)

SKILL BUILDERS
PROOF

A proof convinces an audience that a conjecture is true for ALL cases (situations) that fit the conditions of the conjecture. For example, "If a polygon is a triangle on a flat surface, then the sum of the measures of the angles is 180°." Because we proved this conjecture near the end of Unit 3, it is always true. There are many formats that may be used to write a proof. This course explores three of them, namely, paragraph, flow chart, and two-column.

Example
If $\overline{BD}$ is a perpendicular bisector of $\overline{AC}$, prove that $\triangle ABC$ is isosceles.

Paragraph proof
To prove that $\triangle ABC$ is isosceles, show that $\overline{BA} \cong \overline{BC}$. We can do this by showing that the two segments are corresponding parts of congruent triangles.

Since $\overline{BD}$ is perpendicular to $\overline{AC}$, $m\angle BDA = m\angle BDC = 90°$.

Since $\overline{BD}$ bisects $\overline{AC}$, $\overline{AD} \cong \overline{CD}$. With $\overline{BD} \cong \overline{BD}$ (reflexive property), $\triangle ADB \cong \triangle CDB$ by SAS.

Finally, $\overline{BA} \cong \overline{BC}$ because corresponding parts of congruent triangles are congruent. Therefore, $\triangle ABC$ must be isosceles since two of the three sides are congruent.

Flow chart proof

Two-Column Proof

Given: $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$

$\angle ADB \cong \angle BDC$

$\overline{AD} \cong \overline{CD}$

Bisector

$\overline{CBD} \cong \overline{ABD}$

$\overline{BD} \cong \overline{BD}$

Reflexive

$\overline{BA} \cong \overline{BC}$

SAS

$\angle ADB$ and $\angle BDC$

SAS have parts

$\triangle ABC$ is isosceles

Definition of isosceles

Statement | Reason
--- | ---
$BD$ bisects $\overline{AC}$ | Given
$BD \perp \overline{AC}$ | Given
$\overline{AD} \cong \overline{CD}$ | Def. of bisector
$\angle ADB$ and $\angle BDC$ | Def. of perpendicular
are right angles | All right angles are $\cong$.
$\angle ADB \cong \angle BDC$ | Reflexive property
$\overline{BD} \cong \overline{BD}$ | S.A.S.
$\triangle ABD \cong \triangle CBD$ | $\cong$'s have $\cong$ parts
$\overline{AB} \cong \overline{CB}$ | Def. of isosceles
$\therefore \triangle ABC$ is isosceles | Def. of isosceles
In each diagram below, are any triangles congruent? If so, prove it. (Note: It is good practice to try different methods for writing your proofs.)

1. \[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

2. \[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

3. \[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

4. \[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

5. \[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

6. \[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

Complete a proof for each problem below in the style of your choice.

7. Given: TR and MN bisect each other. Prove: \(\triangle NTP \cong \triangle MRP\)

8. Given: CD bisects \(\angle ACB\); \(\angle 1 \cong \angle 2\). Prove: \(\triangle CDA \cong \triangle CDB\)

9. Given: \(\overline{AB} \parallel \overline{CD}\), \(\angle B \cong \angle D\), \(AB \cong CD\). Prove: \(\triangle ABF \cong \triangle CED\)

10. Given: \(\overline{PG} \cong \overline{SG}\), \(\overline{TP} \cong \overline{TS}\). Prove: \(\triangle TPG \cong \triangle TSG\)

11. Given: OE \perp MP, OE bisects \(\angle MOP\). Prove: \(\triangle MOE \cong \triangle POE\)

12. Given: \(\overline{AD} \parallel \overline{BC}\), \(\overline{DC} \parallel \overline{BA}\). Prove: \(\triangle ADB \cong \triangle CBD\)
13. Given: \(\overline{AC}\) bisects \(\overline{DE}\), \(\angle A \cong \angle C\)
Prove: \(\triangle ADB \cong \triangle CEB\)

14. Given: \(PQ \perp RS\), \(\angle R \cong \angle S\)
Prove: \(\triangle PQR \cong \triangle PQS\)

15. Given: \(\angle S \cong \angle R\), \(\overline{PQ}\) bisects \(\angle SQR\)
Prove: \(\triangle SPQ \cong \triangle RPQ\)

16. Given: \(\overline{TU} \cong \overline{GY}\), \(\overline{KY} || \overline{HU}\), \(\overline{KT} \perp \overline{TG}\), \(\overline{HG} \perp \overline{TG}\)
Prove: \(\angle K \cong \angle H\)

17. Given: \(\overline{MQ} || \overline{WL}\), \(\overline{MQ} \cong \overline{WL}\)
Prove: \(\overline{ML} || \overline{WQ}\)

Consider the diagram below.

18. Is \(\triangle BCD \cong \triangle EDC\)? Prove it!
19. Is \(AB \cong DC\)? Prove it!
20. Is \(AB \cong ED\)? Prove it!

**Answers**

1. Yes
   \[\angle BAD \cong \angle BCD\]
   \[\angle BDC \cong \angle BDA\]
   Given
   \(BD = BD\)
   Reflexive
   \(\angle ABD \cong \angle CBD\)
   AAS

2. Yes
   \[\angle B \cong \angle E\]
   \[\angle BCA \cong \angle BCD\]
   Given
   \(BC = CE\)
   vertical \(\angle s\) are \(\cong\)
   \(\angle ABC \cong \angle DEC\)
   ASA
3. Yes

Given: $\angle BCD \cong \angle BCA$

Right $\angle$s are $\cong$

SAS

4. Yes

Given: $AD \cong BC$

Ca: $\cong$

S

SSS

5. Not necessarily. Counterexample:

6. Yes

5. Given: $\overline{AC} \cong \overline{FD}$

$\angle ABC \cong \angle DEF$

HL

6. Given: $\overline{BA} \cong \overline{CD}$

$\angle ABC \cong \angle CDA$

S

6. Given: $\overline{CA} \cong \overline{CA}$

$\cong$

$\cong$

$\cong$

$\cong$

7. NP $\cong MP$ and $\overline{TP} \cong TP$ by definition of bisector. $\angle NPT \cong \angle MPR$ because vertical angles are equal. So, $\triangle NTP \cong \triangle MRP$ by SAS.

8. $\angle ACD \cong \angle BCD$ by definition of angle bisector. $\overline{CD} \cong \overline{CD}$ by reflexive so $\triangle CDA \cong \triangle CDB$ by ASA.

9. $\angle A \cong \angle C$ since alternate interior angles of parallel lines congruent so $\triangle AFB \cong \triangle CED$ by ASA.

10. $\overline{TG} \cong \overline{TG}$ by reflexive so $\triangle TPG \cong \triangle TSP$ by SAS.

11. $\angle MOE \cong \angle POE$ because perpendicular lines form congruent right angles. $\overline{OE} \cong \overline{OE}$ by reflexive. So, $\triangle MOE \cong \triangle POE$ by ASA.

12. $\angle CDB \cong \angle ABD$ and $\angle ADB \cong \angle CBD$ since parallel lines give congruent alternate interior angles. $\overline{DB} \cong \overline{DB}$ by reflexive so $\triangle ADB \cong \triangle CBD$ by ASA.

13. $\overline{AB} \cong \overline{EB}$ by definition of bisector. $\angle DBA \cong \angle EBC$ since vertical angles are congruent. So $\triangle ADB \cong \triangle CEB$ by AAS.

14. $\overline{RQP} \cong \overline{SQP}$ since perpendicular lines form congruent right angles. $\overline{PQ} \cong \overline{PQ}$ by reflexive so $\triangle PQR \cong \triangle PQS$ by AAS.

15. $\angle SQP \cong \angle RQP$ by angle bisector and $\overline{PQ} \cong \overline{PQ}$ by reflexive, so $\triangle SPQ \cong \triangle RPQ$ by AAS.

16. $\angle KYT \cong \angle HUG$ because parallel lines form congruent alternate exterior angles. $\overline{TY} + \overline{YU} + \overline{GU}$ so $\overline{TY} \cong \overline{GU}$ by subtraction. $\angle T \cong \angle G$ since perpendicular lines form congruent right angles. So $\triangle KTY \cong \triangle HGU$ by ASA. Therefore, $\angle K \cong \angle H$ since $\cong$ triangles have congruent parts.

17. $\angle MQL \cong \angle WLQ$ since parallel lines form congruent alternate interior angles. $\overline{QL} \cong \overline{QL}$ by reflexive so $\triangle MQL \cong \triangle WLQ$ by SAS so $\angle WQL \cong \angle MLQ$ since congruent triangles have congruent parts. So $\overline{ML} || \overline{WQ}$ since congruent alternate interior angles are formed by parallel lines.

18. Yes


20. Not necessarily.
RADICALS

Sometimes it is convenient to leave square roots in radical form instead of using a calculator to find approximations (decimal values). Look for perfect squares (i.e., 4, 9, 16, 25, 36, 49, ...) as factors of the number that is inside the radical sign (radicand) and take the square root of any perfect square factor. Multiply the root of the perfect square times the reduced radical. When there is an existing value that multiplies the radical, multiply any root(s) times that value.

For example:

\[ \sqrt{9} = 3 \quad \text{and} \quad 5\sqrt{9} = 5 \cdot 3 = 15 \]
\[ \sqrt{18} = \sqrt{9} \cdot 2 = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2} \quad \text{and} \quad 3\sqrt{98} = 3\sqrt{49} \cdot 2 = 3 \cdot 7 \sqrt{2} = 21\sqrt{2} \]
\[ \sqrt{80} = \sqrt{16} \cdot 5 = \sqrt{16} \cdot \sqrt{5} = 4\sqrt{5} \quad \text{and} \quad \sqrt{45} + 4\sqrt{20} = \sqrt{9} \cdot 5 + 4\sqrt{4} \cdot 5 = 3\sqrt{5} + 4 \cdot 2 \sqrt{5} = 11\sqrt{5} \]

When there are no more perfect square factors inside the radical sign, the product of the whole number (or fraction) and the remaining radical is said to be in SIMPLE RADICAL FORM.

Simple radical form does not allow radicals in the denominator of a fraction. If there is a radical in the denominator, RATIONALIZE THE DENOMINATOR by multiplying the numerator and denominator of the fraction by the radical in the original denominator. Then simplify the remaining fraction. Examples:

\[ \frac{2}{\sqrt{2}} = \frac{2 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2} \quad \text{and} \quad \frac{4\sqrt{5}}{\sqrt{6}} = \frac{4\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{4\sqrt{30}}{6} = \frac{2\sqrt{30}}{3} \]

In the first example, \( \sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2 \) and \( \frac{2}{2} = 1 \). In the second example, \( \sqrt{6} \cdot \sqrt{6} = \sqrt{36} = 6 \) and \( \frac{4}{6} = \frac{2}{3} \).

Example 1

Add \( \sqrt{27} + \sqrt{12} - \sqrt{48} \). Factor each radical and simplify.

\[ \sqrt{9} \cdot 3 + \sqrt{4} \cdot 3 - \sqrt{16} \cdot 3 = 3\sqrt{3} + 2\sqrt{3} - 4\sqrt{3} = 1\sqrt{3} \quad \text{or} \quad \sqrt{3} \]

Example 2

Simplify \( \frac{3}{\sqrt{6}} \). Multiply by \( \frac{\sqrt{6}}{\sqrt{6}} \) and simplify: \( \frac{3}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2} \).
Write each of the following radicals in simple radical form.

1. $\sqrt{24}$
2. $\sqrt{48}$
3. $\sqrt{7}$
4. $\sqrt{31}$
5. $\sqrt{75}$
6. $\sqrt{50}$
7. $\sqrt{96}$
8. $\sqrt{243}$
9. $\sqrt{8 + \sqrt{18}}$
10. $\sqrt{18 + \sqrt{32}}$
11. $\sqrt{27} - \sqrt{12}$
12. $\sqrt{50} - \sqrt{32}$
13. $\sqrt{6} + \sqrt{63}$
14. $\sqrt{44} + \sqrt{99}$
15. $\sqrt{50} + \sqrt{32} - \sqrt{27}$
16. $\sqrt{75} - \sqrt{8} - \sqrt{32}$
17. $\frac{3}{\sqrt{3}}$
18. $\frac{5}{\sqrt{27}}$
19. $\sqrt{3} - 6\sqrt{2}$
20. $\sqrt{5}$
21. $4\sqrt{5} - \frac{10}{\sqrt{6}}$

Answers

1. $2\sqrt{6}$
2. $4\sqrt{3}$
3. $\sqrt{17}$
4. $\sqrt{31}$
5. $5\sqrt{3}$
6. $5\sqrt{2}$
7. $4\sqrt{6}$
8. $9\sqrt{3}$
9. $5\sqrt{2}$
10. $7\sqrt{2}$
11. $\sqrt{3}$
12. $\sqrt{2}$
13. $3\sqrt{7} + \sqrt{6}$
14. $5\sqrt{11}$
15. $9\sqrt{2} - 3\sqrt{5}$
16. $5\sqrt{3} - 6\sqrt{2}$
17. $\sqrt{3}$
18. $\frac{5\sqrt{3}}{9}$
19. $\frac{1\sqrt{5}}{5}$
20. $\frac{\sqrt{35}}{7}$
21. $2\sqrt{5}$
The three basic trigonometric ratios for right triangles are the sine (pronounced "sign"), cosine, and tangent. Each one is used in separate situations, and the easiest way to remember which to use when is the mnemonic SOH-CAH-TOA. With reference to one of the acute angles in a right triangle, Sine uses the Opposite and the Hypotenuse - SOH. The Cosine uses the Adjacent side and the Hypotenuse - CAH, and the Tangent uses the Opposite side and the Adjacent side - TOA. In each case, the position of the angle determines which leg (side) is opposite or adjacent. Remember that opposite means “across from” and adjacent means “next to.”

\[
\begin{align*}
\text{tan A} &= \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{BC}{AC} \\
\text{sin A} &= \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{BC}{AB} \\
\text{cos A} &= \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{AC}{AB}
\end{align*}
\]
**Example 1**

Use trigonometric ratios to find the lengths of each of the missing sides of the triangle below.

The length of the adjacent side with respect to the 42° angle is 17 ft. To find the length \( y \), use the tangent because \( y \) is the opposite side and we know the adjacent side.

\[
\tan 42° = \frac{y}{17}
\]

\[17 \tan 42° = y\]

\[15.307 \text{ ft} \approx y\]

The length of \( y \) is approximately 15.31 feet.

To find the length \( h \), use the cosine ratio (adjacent and hypotenuse).

\[
\cos 42° = \frac{17}{h}
\]

\[h \cos 42° = 17\]

\[h = \frac{17}{\cos 42°} \approx 22.876 \text{ ft}\]

The hypotenuse is approximately 22.9 feet long.

**Example 2**

Use trigonometric ratios to find the size of each angle and the missing length in the triangle below.

To find \( m\angle u \), use the tangent ratio because you know the opposite (18 ft) and the adjacent (21 ft) sides.

\[
\tan u° = \frac{18}{21}
\]

\[m\angle u° = \tan^{-1} \frac{18}{21} \approx 40.601°\]

The measure of angle \( u \) is approximately 40.6°. By subtraction we know that \( m\angle v \approx 49.4° \).

Use the sine ratio for \( m\angle u \) and the opposite side and hypotenuse.

\[
\sin 40.6° = \frac{18}{h}
\]

\[h \sin 40.6 = 18\]

\[h = \frac{18}{\sin 40.6°} \approx 27.659 \text{ ft}\]

The hypotenuse is approximately 27.7 feet long.
Use trigonometric ratios to solve for the variable in each figure below.

1. \[ \begin{align*}
38 \degree & \quad 15 \\
& \quad h
\end{align*} \]

2. \[ \begin{align*}
26 \degree & \quad 8 \\
& \quad h
\end{align*} \]

3. \[ \begin{align*}
23 & \quad 49 \degree \\
x & \quad 40 \degree
\end{align*} \]

4. \[ \begin{align*}
37 & \quad 41 \degree \\
x & \quad 41 \degree
\end{align*} \]

5. \[ \begin{align*}
y & \quad 15 \degree \\
38 & \quad 15 \degree
\end{align*} \]

6. \[ \begin{align*}
y & \quad 55 \degree \\
43 & \quad 43 \degree
\end{align*} \]

7. \[ \begin{align*}
15 & \quad z \\
z & \quad 38 \degree
\end{align*} \]

8. \[ \begin{align*}
z & \quad 52 \degree \\
18 & \quad 18 \degree
\end{align*} \]

9. \[ \begin{align*}
w & \quad 38 \degree \\
23 & \quad 23 \degree
\end{align*} \]

10. \[ \begin{align*}
w & \quad 38 \degree \\
15 & \quad 15 \degree
\end{align*} \]

11. \[ \begin{align*}
38 & \quad 15 \degree \\
x & \quad 15 \degree
\end{align*} \]

12. \[ \begin{align*}
29 & \quad 38 \degree \\
x & \quad 91
\end{align*} \]

13. \[ \begin{align*}
5 & \quad x \degree \\
7 & \quad 7 \degree
\end{align*} \]

14. \[ \begin{align*}
u & \quad 9 \degree \\
7 & \quad 7 \degree
\end{align*} \]

15. \[ \begin{align*}
v & \quad y \degree \\
12 & \quad 18 \degree
\end{align*} \]

16. \[ \begin{align*}
v \degree & \quad 78 \\
88 & \quad 88
\end{align*} \]

Draw a diagram and use trigonometric ratios to solve each of the following problems.

17. Juanito is flying a kite at the park and realizes that all 500 feet of string are out. Margie measures the angle of the string with the ground with her clinometer and finds it to be \(42.5\degree\). How high is Juanito’s kite above the ground?

18. Nell’s kite has a 350 foot string. When it is completely out, Ian measures the angle to be \(47.5\degree\). How far would Ian need to walk to be directly under the kite?

19. Mayfield High School’s flagpole is 15 feet high. Using a clinometer, Tamara measured an angle of \(11.3\degree\) to the top of the pole. Tamara is 62 inches tall. How far from the flagpole is Tamara standing?
20. Tamara took another sighting of the top of the flagpole from a different position. This time the angle is 58.4°. If everything else is the same, how far from the flagpole is Tamara standing?

Answers

1. \[ h = 15 \sin 38^\circ \approx 9.235 \]
2. \[ h = 8 \sin 26^\circ \approx 3.507 \]
3. \[ x = 23 \cos 49^\circ \approx 15.089 \]
4. \[ x = 37 \cos 41^\circ \approx 27.924 \]
5. \[ y = 38 \tan 15^\circ \approx 10.182 \]
6. \[ y = 43 \tan 55^\circ \approx 61.4104 \]
7. \[ z = \frac{15}{\sin 38^\circ} \approx 24.364 \]
8. \[ z = \frac{18}{\sin 52^\circ} \approx 22.8423 \]
9. \[ w = \frac{23}{\cos 38^\circ} \approx 29.1874 \]
10. \[ w = \frac{15}{\cos 38^\circ} \approx 19.0353 \]
11. \[ x = \frac{38}{\tan 15^\circ} \approx 141.818 \]
12. \[ x = \frac{91}{\tan 29^\circ} \approx 164.168 \]
13. \[ x = \tan^{-1} \frac{5}{7} \approx 35.5377^\circ \]
14. \[ u = \tan^{-1} \frac{7}{9} \approx 37.875^\circ \]
15. \[ y = \tan^{-1} \frac{12}{18} \approx 33.690^\circ \]
16. \[ v = \tan^{-1} \frac{78}{88} \approx 41.5526^\circ \]

17. \[
\sin 42.5^\circ = \frac{h}{500} \\
h = 500 \sin 42.5^\circ \approx 337.795 \text{ ft}
\]

18. \[
\cos 47.5^\circ = \frac{d}{350} \\
d = 350 \cos 47.5^\circ \approx 236.46 \text{ ft}
\]

19. \[
15 \text{ ft} = 180 \text{ inches}, 180" - 62" = 118" = h \\
x = \frac{590.5 \text{ inches or } 49.2 \text{ ft}}{}
\]

20. \[
h = 118", \tan 58.4^\circ = \frac{118"}{x}, \\
x \tan 58.4 = 118", x = \frac{118"}{\tan 58.4^\circ} \\
x \approx 72.59 \text{ inches or } 6.05 \text{ ft}
\]
**RATIO OF SIMILARITY**

The **RATIO OF SIMILARITY** between any two similar figures is the ratio of any pair of corresponding sides. Simply stated, once it is determined that two figures are similar, all of their pairs of corresponding sides have the same ratio.

The ratio of similarity of figure P to figure Q, written P : Q, is \( \frac{3}{5} \).

4.5 : 7.5 and 12 : 20 have the same ratio.

Note that the ratio of similarity is always expressed in lowest possible terms. Also, the order of the statement, P : Q or Q : P, determines which order to state and use the ratios between pairs of corresponding sides.

An Equation that sets one ratio equal to another ratio is called a **proportion**. \( \frac{3}{5} = \frac{12}{20} \).

**Example**

Find \( x \) in the figure. Be consistent in matching corresponding parts of similar figures.

\[ \triangle ABC \sim \triangle DEC \text{ by AA (} \angle BAC \cong \angle EDC \text{ and } \angle C \cong \angle C) \].

Therefore the corresponding sides are proportional.

\[
\frac{DE}{AB} = \frac{CD}{CA} \implies \frac{22}{x} = \frac{24}{36} \implies 24x = 22(36) \implies 24x = 792 \implies x = 33
\]

Note: This problem also could have been solved with the proportion: \( \frac{DE}{AB} = \frac{CE}{CB} \).

For each pair of similar figures below, find the ratio of similarity, for large:small.

1. 2. 3.

4. 5. 6.
For each pair of similar figures below, state the ratio of similarity, then use it to find x.

7. \[ \frac{4}{28} = \frac{x}{16} \]

8. \[ \frac{24}{36} = \frac{x}{12} \]

9. \[ \frac{6}{45} = \frac{x}{18} \]

10. \[ \frac{x}{20} = \frac{18}{12} \]

11. \[ \frac{16}{4} = \frac{x}{5} \]

12. \[ \frac{10}{12} = \frac{18}{15} \]

For problems 13 through 18, use the given information and the figure to find each length.

13. JM = 14, MK = 7, JN = 10 Find NL.

14. MN = 5, JN = 4, JL = 10 Find KL.

15. KL = 10, MK = 2, JM = 6 Find MN.

16. MN = 5, KL = 10, JN = 7 Find JL.

17. JN = 3, NL = 7, JM = 5 Find JK.

18. JK = 37, NL = 7, JM = 5 Find JN.

19. Standing 4 feet from a mirror laying on the flat ground, Palmer, whose eye height is 5 feet, 9 inches, can see the reflection of the top of a tree. He measures the mirror to be 24 feet from the tree. How tall is the tree?

20. The shadow of a statue is 20 feet long, while the shadow of a student is 4 ft long. If the student is 6 ft tall, how tall is the statue?

Answers

1. \[ \frac{4}{3} \]
2. \[ \frac{5}{1} \]
3. \[ \frac{2}{1} \]
4. \[ \frac{24}{7} \]
5. \[ \frac{6}{1} \]
6. \[ \frac{15}{8} \]
7. \[ \frac{7}{8}; x = 32 \]
8. \[ \frac{2}{1}; x = 72 \]
9. \[ \frac{1}{3}; x = 15 \]
10. \[ \frac{5}{6}; x = 15 \]
11. \[ \frac{4}{5}; x = 20 \]
12. \[ \frac{3}{7}; x = 16.5 \]
13. 5
14. 12.5
15. 7.5
16. 14
17. 16 \[ \frac{2}{3} \]
18. 11 \[ \frac{25}{32} \]
19. 34.5 ft.
20. 30 ft.

SKILL BUILDERS
SIMILARITY OF LENGTH, AREA, AND VOLUME

The relationships for similarity of length, area, and volume are developed in Unit 8 and summarized in problem S-68 on page 322. The basic relationships of the \( r: r^2 : r^3 \) Theorem are stated below.

Once you know two figures are similar with a ratio of similarity \( \frac{a}{b} \), the following proportions for the SMALL (sm) and LARGE (lg) figures (which are enlargements or reductions of each other) are true:

\[
\frac{\text{side}_{\text{sm}}}{\text{side}_{\text{lg}}} = \frac{a}{b} \quad \frac{P_{\text{sm}}}{P_{\text{lg}}} = \frac{a}{b} \quad \frac{A_{\text{sm}}}{A_{\text{lg}}} = \frac{a^2}{b^2} \quad \frac{V_{\text{sm}}}{V_{\text{lg}}} = \frac{a^3}{b^3}
\]

In each proportion above, the data from the smaller figure is written on top (in the numerator) to help be consistent with correspondences. When working with area, the basic ratio of similarity is squared; for volume, it is cubed. NEVER square or cube the actual areas or volumes themselves.

Example 1

The two rectangular prisms above are similar. Suppose the ratio of their vertical edges is 3:8. Use the \( r: r^2 : r^3 \) Theorem to find the following without knowing the dimensions of the prisms.

a) Find the ratio of their surface areas.

b) Find the ratio of their volumes.

c) The perimeter of the front face of the large prism is 18 units. Find the perimeter of the front face of the small prism.

d) The area of the front face of the large prism is 15 square units. Find the area of the front face of the small prism.

e) The volume of the small prism is 21 cubic units. Find the volume of the large prism.

Use the ratio of similarity for parts (a) and (b): the ratio of the surface areas is \( \frac{3^2}{8^2} = \frac{9}{64} \); the ratio of the volumes is \( \frac{3^3}{8^3} = \frac{27}{512} \). Use proportions to solve for the rest of the parts.

\[
\begin{align*}
8P_{\text{lg}} &= 54 \\
P_{\text{sm}} &= 6.75 \text{ units} \\
\frac{9}{64} &= \frac{\text{A}_{\text{sm}}}{15} \\
\frac{27}{512} &= \frac{\text{V}_{\text{sm}}}{\text{V}_{\text{lg}}} \\
64\text{A}_{\text{sm}} &= 135 \\
27\text{V}_{\text{sm}} &= 10752 \\
\text{A} &\approx 2.11 \text{ sq. units} \\
\text{V} &\approx 398.22 \text{ cu. units}
\end{align*}
\]
Solve each of the following problems.

1. Two rectangular prisms are similar. The smaller, A, has a height of four units while the larger, B, has a height of six units.
   a) What is the magnification factor from prism A to prism B?
   b) What would be the ratio of the lengths of the edges labeled $x$ : $y$?
   c) What is the ratio, small to large, of their surface areas? their volumes?
   d) A third prism, C is similar to prisms A and B. Prism C's height is ten units. If the volume of prism A is 24 cubic units, what is the volume of prism C?

2. If rectangle A and rectangle B have a ratio of similarity of 5:4, what is the area of rectangle B if the area of rectangle A is 24 square units?

3. If rectangle A and rectangle B have a ratio of similarity of 2:3, what is the area of rectangle B if the area of rectangle A is 46 square units?

4. If rectangle A and rectangle B have a ratio of similarity of 3:4, what is the area of rectangle B if the area of rectangle A is 82 square units?

5. If rectangle A and rectangle B have a ratio of similarity of 1:5, what is the area of rectangle B if the area of rectangle A is 24 square units?

6. Rectangle A is similar to rectangle B. The area of rectangle A is 81 square units while the area of rectangle B is 49 square units. What is the ratio of similarity between the two rectangles?

7. Rectangle A is similar to rectangle B. The area of rectangle B is 18 square units while the area of rectangle A is 12.5 square units. What is the ratio of similarity between the two rectangles?

8. Rectangle A is similar to rectangle B. The area of rectangle A is 16 square units while the area of rectangle B is 100 square units. If the perimeter of rectangle A is 12 units, what is the perimeter of rectangle B?

9. If prism A and prism B have a ratio of similarity of 2:3, what is the volume of prism B if the volume of prism A is 36 cubic units?

10. If prism A and prism B have a ratio of similarity of 1:4, what is the volume of prism B if the volume of prism A is 83 cubic units?

11. If prism A and prism B have a ratio of similarity of 6:11, what is the volume of prism B if the volume of prism A is 96 cubic units?
12. Prism A and prism B are similar. The volume of prism A is 72 cubic units while the volume of prism B is 1125 cubic units. What is the ratio of similarity between these two prisms?

13. Prism A and prism B are similar. The volume of prism A is 27 cubic units while the volume of prism B is approximately 512 cubic units. If the surface area of prism B is 128 square units, what is the surface area of prism A?

14. The corresponding diagonals of two similar trapezoids are in the ratio of 1:7. What is the ratio of their areas?

15. The ratio of the perimeters of two similar parallelograms is 3:7. What is the ratio of their areas?

16. The ratio of the areas of two similar trapezoids is 1:9. What is the ratio of their altitudes?

17. The areas of two circles are in the ratio of 25:16. What is the ratio of their radii?

18. The ratio of the volumes of two similar circular cylinders is 27:64. What is the ratio of the diameters of their similar bases?

19. The surface areas of two cubes are in the ratio of 49:81. What is the ratio of their volumes?

20. The ratio of the weights of two spherical steel balls is 8:27. What is the ratio of the diameters of the two steel balls?
### Answers

1. a) $\frac{6}{4} = \frac{3}{2}$  
   b) $\frac{4}{6} = \frac{2}{3}$  
   c) $\frac{16}{36}$ or $\frac{4}{9}$  
   d) $\frac{64}{216}$ or $\frac{16}{27}$  
   d) $375$ cu unit

2. $15.36$ u²

3. $103.5$ u²

4. $\approx 145.8$ u²

5. $600$ u²

6. $\frac{9}{7}$

7. $\frac{6}{5}$

8. $30$ u

9. $121.5$

10. $5312$

11. $\approx 591.6$

12. $\frac{2}{5}$

13. $\approx 18$ u²

14. $\frac{1}{49}$

15. $\frac{9}{49}$

16. $\frac{1}{3}$

17. $\frac{5}{4}$

18. $\frac{3}{4}$

19. $\frac{343}{729}$

20. $\frac{2}{3}$
INTRODUCTION AND EXTERIOR ANGLES OF POLYGONS #18

The sum of the measures of the interior angles of an n-gon is \( \text{sum} = (n - 2)180^\circ \).

The measure of each angle in a regular n-gon is \( m\angle = \frac{(n-2)180^\circ}{n} \).

The sum of the exterior angles of any n-gon is 360°.

**Example 1**
Find the sum of the interior angles of a 22-gon.

Since the polygon has 22 sides, we can substitute this number for \( n \):

\[
\text{sum} = (n - 2)180^\circ = (22 - 2)180^\circ = 20 \cdot 180^\circ = 3600^\circ.
\]

**Example 2**
If the 22-gon is regular, what is the measure of each angle? Use the sum from the previous example and divide by 22:

\[
3600^\circ \div 22 \approx 163.64^\circ.
\]

**Example 3**
Each angle of a regular polygon measures 157.5°. How many sides does this n-gon have?

a) Solving algebraically:

\[
157.5^\circ = \frac{(n-2)180^\circ}{n} \Rightarrow 157.5n = (n - 2)180^\circ \Rightarrow 157.5n = 180n - 360 \Rightarrow -22.5n = -360 \Rightarrow n = 16
\]

b) If each interior angle is 157.5°, then each exterior angle is 180° - 157.5° = 22.5°.

Since the sum of the exterior angles of any n-gon is 360°, 360° ÷ 22.5° = 16 sides.

**Example 4**
Find the area of a regular 7-gon with sides of length 5 ft.

Because the regular 7-gon is made up of 7 identical, isosceles triangles we need to find the area of one, and then multiply it by 7. (See page 349 for addition figures and details that parallel this solution.) In order to find the area of each triangle we need to start with the angles of each triangle.

Each interior angle of the regular 7-gon measures 157.5°. The angle in the triangle is half the size of the interior angle, so \( m\angle \approx \frac{128.57^\circ}{2} \approx 64.29^\circ \). Find the height of the triangle by using the tangent ratio:

\[
\tan \angle 1 = \frac{h}{2.5} \Rightarrow h = 2.5 \cdot \tan \angle 1 \approx 5.19 \text{ ft}.
\]

Thus the area of the 7-gon is \( 7 \cdot 12.98 \approx 90.86 \text{ ft}^2 \).
Find the measures of the angles in each problem below.

1. Find the sum of the interior angles in a 7-gon.
2. Find the sum of the interior angles in an 8-gon.
3. Find the size of each of the interior angles in a regular 12-gon.
4. Find the size of each of the interior angles in a regular 15-gon.
5. Find the size of each of the exterior angles of a regular 17-gon.
6. Find the size of each of the exterior angles of a regular 21-gon.

Solve for x in each of the figures below.

7. 
8. 
9. 
10. 

Answer each of the following questions.

11. Each exterior angle of a regular n-gon measures \(16\frac{4}{11}\)°. How many sides does this n-gon have?
12. Each exterior angle of a regular n-gon measures \(13\frac{1}{3}\)°. How many sides does this n-gon have?
13. Each angle of a regular n-gon measures 156°. How many sides does this n-gon have?
14. Each angle of a regular n-gon measures 165.6°. How many sides does this n-gon have?
15. Find the area of a regular pentagon with side length 8 cm.
16. Find the area of a regular hexagon with side length 10 ft.
17. Find the area of a regular octagon with side length 12 m.
18. Find the area of a regular decagon with side length 14 in.

**Answers**

1. 900°
2. 1080°
3. 150°
4. 156°
5. 21.1765°
6. 17.1429°
7. \(x = 24°\)
8. \(x = 30°\)
9. \(x = 98.18°\)
10. \(x = 31.30°\)
11. 22 sides
12. 27 sides
13. 15 sides
14. 25 sides
15. \(110.1106\) cm²
16. 259.8076 ft$^2$  
17. 695.2935 m$^2$  
18. 1508.0649 in$^2$
AREAS BY DISSECTION

DISSECTION PRINCIPLE: Every polygon can be dissected (or broken up) into triangles which have no interior points in common. This principle is an example of the problem solving strategy of subproblems. Finding simpler problems that you know how to solve will help you solve the larger problem.

Example
Find the area of the figure below.

```
2 4 2 4
2 3 3 2
6 2
```

First fill in the missing lengths. The right vertical side of the figure is a total of 8 units tall. Since the left side is also 8 units tall, \(8 - 4 - 2 = 2\) units. The missing vertical length on the top middle cutout is also 2.

Horizontally, the bottom length is 8 units \((6 + 2)\). Therefore, the top is 2 units \((8 - 4 - 2)\). The horizontal length at the left middle is 3 units.

Enclose the entire figure in an 8 by 8 square and then remove the area of the three rectangles that are not part of the figure. The area is:

- Enclosing square: \(8 \times 8 = 64\)
- Top cut out: \(2 \times 2 = 4\)
- Left cut out: \(3 \times 2 = 6\)
- Lower right cut out: \(2 \times 3 = 6\)

Area of figure: \(64 - 4 - 6 - 6 = 48\)

Example
Find the area of the figure below.

This figure consists of five familiar figures: a central square, 5 units by 5 units; three triangles, one on top with \(b = 5\) and \(h = 3\), one on the right with \(b = 4\) and \(h = 3\), and one on the bottom with \(b = 5\) and \(h = 2\); and a trapezoid with an upper base of 4, a lower base of 2 and a height of 2.

The area is:

- square: \(5 \times 5 = 25\)
- top triangle: \(\frac{1}{2} \times 5 \times 3 = 7.5\)
- bottom triangle: \(\frac{1}{2} \times 5 \times 2 = 5\)
- right triangle: \(\frac{1}{2} \times 3 \times 4 = 6\)
- trapezoid: \(\frac{(4 + 2) \times 2}{2} = 6\)

Total area: \(49.5 \text{ u}^2\)

Find the area of each of the following figures. Assume that anything that looks like a right angle is a right angle.
1. \[ \text{Area} = 42 \text{ u}^2 \]
2. \[ \text{Area} = 33 \text{ u}^2 \]
3. \[ \text{Area} = 85 \text{ u}^2 \]
4. \[ \text{Area} = 31 \text{ u}^2 \]
5. \[ \text{Area} = 36 \text{ u}^2 \]
6. \[ \text{Area} = 36 \text{ u}^2 \]
7. \[ \text{Area} = 36 \text{ u}^2 \]
8. \[ \text{Area} = 29.5 \text{ u}^2 \]
9. \[ \text{Area} = 46 \text{ u}^2 \]
10. \[ \text{Area} = 28 \text{ u}^2 \]

11. \[ \text{SA} = 196 \text{ sq.un.} \]
12. \[ \text{Area} = 312 \text{ sq.un.} \]
13. \[ A = 32u^2 \quad \text{P} = \sqrt{34} + 3 \sqrt{2} + 12 \approx 38.3162u \]

14. \[ A = 41.5u^2 \quad \text{P} = 18 + 4 \sqrt{2} + \sqrt{10} \approx 26.8191u \]

15. \[ A = 21u^2 \quad \text{P} = 16 + 6 \sqrt{2} \approx 24.4853u \]

16. \[ A = 14 + 6 \sqrt{3} \approx 24.3923u^2 \quad \text{P} = 24 + 2 \sqrt{6} \approx 28.899u \]
CENTRAL AND INSCRIBED ANGLES

A central angle is an angle whose vertex is the center of a circle and whose sides intersect the circle. The degree measure of a central angle is equal to the degree measure of its intercepted arc. For the circle at right with center C, $\angle ACB$ is a central angle.

An INSCRIBED ANGLE is an angle with its vertex on the circle and whose sides intersect the circle. The arc formed by the intersection of the two sides of the angle and the circle is called an INTERCEPTED ARC. $\angle ADB$ is an inscribed angle, $AB$ is an intercepted arc.

The INSCRIBED ANGLE THEOREM says that the measure of any inscribed angle is half the measure of its intercepted arc. Likewise, any intercepted arc is twice the measure of any inscribed angle whose sides pass through the endpoints of the arc.

$$m\angle ADB = \frac{1}{2} AB \quad \text{and} \quad AB = 2m\angle ADB$$
Example 1

In \( \odot P \), \( \angle CPQ = 70^\circ \). Find and \( m \angle CRQ \).

\[
m_{\angle CRQ} = m_{\angle CPQ} = 70^\circ
\]

Example 2

In the circle shown below, the vertex of the angles is at the center. Find \( x \).

Since \( AB \) is the diameter of the circle,
\[ x + 2x = 180^\circ \Rightarrow 3x = 180^\circ \Rightarrow x = 60^\circ \]

Example 3
Solve for $x$.

Since $m\angle BAC = 220^\circ$, $m\angle BC = 360^\circ - 220^\circ = 140^\circ$. The $m\angle BAC$ equals half the $m\angle BC$ or 70°.
Find each measure in $\bigcirc P$ if $m\angle WPX = 28^\circ$, $m\angle ZPY = 38^\circ$, and $\overline{WZ}$ and $\overline{XV}$ are diameters.

1. $m\angle WPX = 28^\circ$
2. $m\angle ZPY = 38^\circ$
3. $\overline{WZ}$
4. $\overline{XV}$
5. $\overline{VPZ}$
6. $\overline{XVP}$
7. $\overline{XWY}$
8. $\overline{WZX}$

In each of the following figures, O is the center of the circle. Calculate the values of $x$ and justify your answer.

9. $x = 136^\circ$
10. $x = 146^\circ$
11. $x = 49^\circ$
12. $x = 62^\circ$
13. $x = 100^\circ$
14. $x = 150^\circ$
15. $x = 18^\circ$
16. $x = 27^\circ$
17. $x = 55^\circ$
18. $x = 91^\circ$
19. $x = 86^\circ$
20. $x = 129^\circ$

**Answers**

1. $38^\circ$
2. $28^\circ$
3. $28^\circ$
4. $180^\circ$
5. $114^\circ$
6. $114^\circ$
7. $246^\circ$
8. $332^\circ$
9. $68^\circ$
10. $73^\circ$
11. $98^\circ$
12. $124^\circ$
13. $50^\circ$
14. $55^\circ$
15. $18^\circ$
<p>| | | | | | |</p>
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<tr>
<td>16</td>
<td>27°</td>
<td>17</td>
<td>55°</td>
<td>18</td>
<td>77°</td>
</tr>
<tr>
<td>19</td>
<td>35°</td>
<td>20</td>
<td>50°</td>
<td></td>
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</table>
**AREA OF SECTORS**

**SECTORS** of a circle are formed by the two radii of a central angle and the arc between their endpoints on the circle. For example, a 60° sector looks like a slice of pizza. See page 376, problem CS-16 for more details and examples.

Example 1

Find the area of the 45° sector. First find the fractional part of the circle involved. \[
\frac{m \angle AB}{360°} = \frac{45°}{360°} = \frac{1}{8}
\]
of the circle's area. Next find the area of the circle: \[A = \pi r^2 = \pi 4^2 = 16\pi \text{ sq. ft.} \]
Finally, multiply the two results to find the area of the sector. \[
\frac{1}{8} \cdot 16\pi = 2\pi \approx 6.28\text{ sq. ft.}
\]

Example 2

Find the area of the 150° sector.

Fractional part of circle is \[
\frac{m \angle AB}{360°} = \frac{150°}{360°} = \frac{5}{12}
\]
Area of circle is \[A = \pi r^2 = \pi 8^2 = 64\pi \text{ sq. ft.} \]
Area of the sector: \[
\frac{5}{12} \cdot 64\pi = \frac{320\pi}{12} = \frac{80\pi}{3} \approx 83.76\text{ sq. ft.}
\]

1. Find the area of the shaded sector in each circle below. Points A, B and C are the centers.
   a) ![Diagram of circle with shaded sector]
   b) ![Diagram of circle with shaded sector]
   c) ![Diagram of circle with shaded sector]

Calculate the area of the following shaded sectors. Point O is the center of each circle.
6. The shaded region in the figure is called a segment of the circle. It can be found by subtracting the area of ΔMIL from the sector MIL. Find the area of the segment of the circle.
Find the area of the shaded regions.

7. \[ \text{Area} = \text{Area of the circle} - \text{Area of the triangle} \]

8. \[ \text{Area} = \text{Area of the cross} - \text{Area of the diamond} \]

9. YARD is a square; A and D are the centers of the arcs.

10. Find the area of a circular garden if the diameter of the garden is 30 feet.

11. Find the area of a circle inscribed in a square whose diagonal is 8 feet long.

12. The area of a 60° sector of a circle is \(10\pi \text{ m}^2\). Find the radius of the circle.

13. The area of a sector of a circle with a radius of 5 mm is \(10\pi \text{ mm}^2\). Find the measure of its central angle.

Find the area of each shaded region.

14. \[ \text{Area} = 10 \times 10 - \frac{60\pi}{60} \]

15. \[ \text{Area} = 14 \times 14 - \frac{60\pi}{60} \]

16. \[ \text{Area} = \frac{363\pi}{4} \]

17. \[ \text{Area} = 8 \pi - \frac{9\pi}{16} \]

18. \[ \text{Area} = \frac{65\pi}{2} \]

19. \[ \text{Area} = \frac{196 - 49\pi}{12} \]

20. Find the radius. The shaded area is \(12\pi \text{ cm}^2\).

Answers

1. a) \(2\pi \text{ un}^2\)  
b) \(\frac{49}{3}\pi \text{ un}^2\)  
c) \(\frac{363\pi}{4} \text{ un}^2\)

2. \(\frac{\pi}{2}\)  

3. \(12\pi\)

4. \(\frac{93\pi}{16}\)

5. \(5\pi\)

6. \(\pi - 2 \text{ sq. un.}\)

7. \(\frac{100\pi}{3} - 25\sqrt{3} \text{ un}^2\)

8. \(10\pi - 20 \text{ sq. un.}\)

9. \(8\pi - 16 \text{ sq. un.}\)

10. \(225\pi \text{ ft}^2\)

11. \(8\pi \text{ ft}^2\)

12. \(2\sqrt{15} \text{ m}\)

13. \(144\)

14. \(100 - \frac{25}{3} \pi \text{ m}^3\)

15. \(196 - 49\pi \text{ ft}^2\)

16. \(10\pi \text{ in}^2\)

17. \(48\pi + 32 \text{ un}^2\)

18. \(\frac{65\pi}{2} \text{ un}^2\)

19. \(\approx 61.8 \text{ un}^2\)

20. \(6 \text{ un.}\)
TANGENTS, SECANTS, AND CHORDS

The figure at right shows a circle with three lines lying on a flat surface. Line a does not intersect the circle at all. Line b intersects the circle in two points and is called a SECANT. Line c intersects the circle in only one point and is called a TANGENT to the circle.

TANGENT/RADIUS THEOREMS:
1. Any tangent of a circle is perpendicular to a radius of the circle at their point of intersection.
2. Any pair of tangents drawn at the endpoints of a diameter are parallel to each other.

A CHORD of a circle is a line segment with its endpoints on the circle.

DIAMETER/CHORD THEOREMS:
1. If a diameter bisects a chord, then it is perpendicular to the chord.
2. If a diameter is perpendicular to a chord, then it bisects the chord.

ANGLE-CHORD-SECANT THEOREMS:
\[ m\angle 1 = \text{Error!} \]
\[ AE \cdot EC = DE \cdot EB \]
\[ m\angle P = \text{Error!} \]
\[ PQ \cdot PR = PS \cdot PT \]
Example 1

If the radius of the circle is 5 units and AC = 13 units, find AD and AB.

AD \perp CD and AB \perp CD by Tangent/Radius Theorem, so (AD)^2 + (CD)^2 = (AC)^2 or (AD)^2 + (5)^2 = (13)^2. So AD = 12 and AB \cong AD so AB = 12.

Example 2

In \( \odot B \), EC = 8 and AB = 5. Find BF.

The diameter is perpendicular to the chord, therefore it bisects the chord, so EF = 4.

AB is a radius and AB = 5. EB is a radius, so EB = 5. Use the Pythagorean Theorem to find BF: BF^2 + 4^2 = 5^2, BF = 3.
In each circle, C is the center and \( \overline{AB} \) is tangent to the circle at point B. Find the area of each circle.

1. \( AC = 30 \)
2. \( AC = 45 \)
3. \( AC = 90 \)
4. \( AC = 16 \)
5. \( AC = 86 \)
6. \( AC = 56 \)
7. \( AC = 24 \)

9. In the figure at right, point E is the center and \( \angle CED = 55^\circ \). What is the area of the circle?

In the following problems, B is the center of the circle. Find the length of \( BF \) given the lengths below.

10. \( EC = 14, AB = 16 \)
11. \( EC = 35, AB = 21 \)
12. \( FD = 5, EF = 10 \)
13. \( EF = 9, FD = 6 \)

14. In \( \odot R \), if \( AB = 2x - 7 \) and \( CD = 5x - 22 \), find \( x \).
15. In \( \odot O \), \( MN \cong \overline{PQ} \), \( MN = 7x + 13 \), and \( PQ = 10x - 8 \). Find \( PS \).
16. In \( \odot D \), if \( AD = 5 \) and \( TB = 2 \), find \( AT \).
17. In $\odot J$, radius JL and chord MN have lengths of 10 cm. Find the distance from J to MN.

18. In $\odot Q$, QC = 13 and QT = 5. Find AB.

19. If $\overline{AC}$ is tangent to circle E and $\overline{EH} \perp \overline{GI}$, is $\triangle GEH \sim \triangle AEB$? Prove your answer.

20. If $\overline{EH}$ bisects $\overline{GI}$ and $\overline{AC}$ is tangent to circle E at point B, are $\overline{AC}$ and $\overline{GI}$ parallel? Prove your answer.

Find the value of $x$.

21. $\overline{AB}$

22. $\overline{BC}$

23. $\overline{CD}$

24. $\overline{DE}$

In $\odot F$, $m_{\overline{AB}} = 84$, $m_{\overline{BC}} = 38$, $m_{\overline{CD}} = 64$, and $m_{\overline{DE}} = 60$. Find the measure of each angle and arc.
25. \( m \angle E A \)
26. \( m \angle A E B \)

27. \( m \angle 1 \)
28. \( m \angle 2 \)
29. \( m \angle 3 \)
30. \( m \angle 4 \)

For each circle, tangent segments are shown. Use the measurements given find the value of \( x \).

31. \(\angle 141^\circ \)
32. \(\angle 70^\circ \)
33. \(\angle 38^\circ \)
34. \(\angle \)
Find each value of \( x \). Tangent segments are shown in problems 40, 43, 46, and 48.

37. \( x \)

38. \( 11 \)

39. \( 2 \)

40. \( 10 \)

41. \( 6 \)

42. \( 8 \)

43. \( 1.0 \)

44. \( 5 \)

45. \( 10 \)

46. \( 4 \)

47. \( 3 \)

48. \( 5 \)

Answers

1. \( 275\pi \) sq. un.
2. \( 1881\pi \) sq. un.
3. \( 36\pi \) sq. un.
4. \( 324\pi \) sq. un.
5. \( 112\pi \) sq. un.
6. \( 4260\pi \) sq. un.
7. \( 7316\pi \) sq. un.
8. \( 49\pi \) sq. un.
9. \( \approx 117.047 \) sq. un.
10. \( \approx 14.4 \)
11. \( \approx 11.6 \)
12. \( \approx 7.5 \)
13. 3.75
14. 5
15. 31
16. 4
17. \( 5\sqrt{3} \) cm.
18. \( 5\sqrt{3} \)
19. Yes, \( \angle \text{GEH} \cong \angle \text{AEB} \) (reflexive). \( \overline{EB} \) is perpendicular to \( \overline{AC} \) since it is tangent so \( \angle \text{GHE} \cong \angle \text{ABE} \) because all right angles are congruent. So the triangles are similar by AA~.
20. Yes. Since \( \overline{EH} \) bisects \( \overline{GI} \) it is also perpendicular to it (SSS). Since \( \overline{AC} \) is a tangent, \( \angle \text{ABE} \) is a right angle. So the lines are parallel since the corresponding angles are right angles and all right angles are equal.

21. 160
22. 9
23. 42
24. 70
25. 114
26. 276
27. 87
28. 49
29. 131
30. 38
31. 40
32. 55
33. 64
34. 38
35. 45
36. 22.5
37. 12
38. \( 5\frac{1}{2} \)
39. 2
40. 30
41. 2
42. \( 2\sqrt{2} \)
43. 1.2
44. 5
45. \( \sqrt{50} \)
46. 6
47. 7.5
48. 5
VOLUME AND SURFACE AREA OF POLYHEDRA

The volume of various polyhedra, that is, the number of cubic units needed to fill each one, is found by using the formulas below.

For prisms and cylinders:
\[ V = \text{base area} \times \text{height}, \quad V = Bh \]

For pyramids and cones:
\[ V = \frac{1}{3} \text{base area} \times \text{height}, \quad V = \frac{1}{3} Bh \]

In prisms and cylinders, you may use either base, since they are congruent. Since the bases of cylinders and cones are circles, their area formulas may be expressed as:

\[ \text{cylinder } V = \pi r^2 h \quad \text{and} \quad \text{cone } V = \frac{1}{3} \pi r^2 h. \]

The surface area of a polyhedron is the sum of the areas of its base(s) and faces.

Example 1

Use the appropriate formula(s) to find the volume of each figure below:

a) This is a triangular pyramid. The base is a right triangle so the area of the base is \[ B = \frac{1}{2} \cdot 8 \cdot 5 = 20 \text{ square units}, \text{ so } V = \frac{1}{3} (20)(22) \approx 146.7 \text{ cubic feet}. \]

b) This is a cylinder. The base is a circle, so \[ B = \pi 5^2, \text{ and } V = (25\pi)(8) = 200\pi \approx 628.32 \text{ cubic feet}. \]

c) This is a cone. The base is a circle, so \[ B = \pi 8^2. \] \[ V = \frac{1}{3} (64\pi)(18) = \frac{1}{3} (64)(18)\pi \]
\[ \Rightarrow = \frac{1}{3} (1152)\pi = 384\pi \approx 1206.37 \text{ feet}^3 \]

d) This prism has a trapezoidal base, so \[ B = \frac{1}{2} (12 + 8)(15) = 150. \]

Thus, \[ V = (150)(14) = 2100 \text{ cubic feet}. \]

This figure is for Example 2, on next page:
Example 2
Find the surface area of the triangular prism shown on the bottom of the previous page. The figure is made up of two triangles (the top and bottom) and three rectangles as shown at right. Find the area of each of these shapes. To find the area of the triangle and the last rectangle, use the Pythagorean Theorem to find the length of the second leg of the right triangular base. Since
\[3^2 + \text{leg}^2 = 5^2, \text{leg} = 4.\]

Calculate all of the areas, and find their sum.
\[SA = 2 \left( \frac{3}{2} \times 4 \right) + 3(8) + 5(8) + 4(8) = 12 + 24 + 40 + 32 = 108 \text{ square units}\]

Example 3
Find the total surface area of a regular square pyramid with a slant height of 10 inches and a base with sides 8 inches long.

The figure is made up of 4 identical triangles and a square base.

Find the volume of each figure.
1.  
2.  
3.  
4.  
5.  
6.
7. Find the volume of the solid shown.

8. Find the volume of the remaining solid after a hole with a diameter of 4 mm is drilled through it.

Find the total surface area of the figures in the previous volume problems.

21. Problem 1  
22. Problem 2  
23. Problem 3  
24. Problem 5  
25. Problem 6  
26. Problem 7  
27. Problem 8  
28. Problem 9  
29. Problem 10  
30. Problem 14  
31. Problem 18  
32. Problem 19

Answers

1. 48 m$^3$  
2. 540 cm$^3$  
3. 14966.6 ft$^3$  
4. 76.9 m$^3$  
5. 1508.75 m$^3$

6. 157 m$^3$  
7. 72 ft$^3$  
8. 1045.4 cm$^3$  
9. 332.6 cm$^3$  
10. 320 in$^3$
11. \(314.2 \text{ in}^3\)  
12. \(609.7 \text{ cm}^3\)  
13. \(2.5 \text{ m}^3\)  
14. \(512 \text{ m}^3\)  
15. \(514.4 \text{ m}^3\)  
16. \(52.3 \text{ cm}^3\)  
17. \(20.9 \text{ cm}^3\)  
18. \(149.3 \text{ in}^3\)  
19. \(7245 \text{ ft}^3\)  
20. \(1011.6 \text{ mm}^3\)  
21. \(80 \text{ m}^2\)  
22. \(468 \text{ cm}^2\)  
23. \(3997.33 \text{ ft}^2\)  
24. \(727.98 \text{ m}^2\)  
25. \(50\pi + 20\pi \approx 219.8 \text{ m}^2\)  
26. \(124 \text{ ft}^2\)  
27. \(121\pi + 189.97 \approx 569.91 \text{ cm}^2\)  
28. \(192 + 48\sqrt{3} \approx 275.14 \text{ cm}^2\)  
29. \(213.21 \text{ in}^2\)  
30. \(576 \text{ in}^2\)  
31. \(193.0 \text{ in}^2\)  
32. \(2394.69 \text{ ft}^2\)
SIMPLIFYING RATIONAL EXPRESSIONS

RATIONAL EXPRESSIONS are fractions that have algebraic expressions in their numerators and/or denominators. To simplify rational expressions find factors in the numerator and denominator that are the same and then write them as fractions equal to 1. For example,

\[
\frac{6}{6} = 1 \quad \frac{x^2}{x^2} = \frac{1(x + 2)}{(x + 2)} = 1 \quad \frac{(3x - 2)}{(3x - 2)} = 1
\]

Notice that the last two examples involved binomial sums and differences. Only when sums or differences are exactly the same does the fraction equal 1. Rational expressions such as the examples below CANNOT be simplified:

\[
\frac{(6 + 5)}{6} \quad \frac{x^3 + y}{x^3} \quad \frac{x}{x + 2} \quad \frac{3x - 2}{2}
\]

Most problems that involve rational expressions will require that you factor the numerator and denominator. For example:

\[
\frac{12}{54} = \frac{2\cdot 2 \cdot 3}{2 \cdot 3 \cdot 3 \cdot 3} = \frac{2}{9} \quad \text{Notice that } \frac{2}{2} \text{ and } \frac{3}{3} \text{ each equal 1.}
\]

\[
\frac{6x^3 y^2}{15x^2 y^4} = \frac{2 \cdot 3 \cdot x^2 \cdot x \cdot y \cdot y^2}{5 \cdot 3 \cdot x^2 \cdot y^2 \cdot y^2} = \frac{2x}{5y^2} \quad \text{Notice that } \frac{3}{3}, \frac{x^2}{x^2}, \text{ and } \frac{y^2}{y^2} = 1.
\]

\[
\frac{x^2 - x - 6}{x^2 - 5x + 6} = \frac{(x + 2)(x - 3)}{(x - 2)(x - 3)} = \frac{x + 2}{x - 2} \quad \text{where } \frac{x - 3}{x - 3} = 1.
\]

All three examples demonstrate that all parts of the numerator and denominator--whether constants, monomials, binomials, or factorable trinomials--must be written as products before you can look for factors that equal 1.

One special situation is shown in the following examples:

\[
\frac{-2}{2} = -1 \quad \frac{-x}{x} = -1 \quad \frac{-x - 2}{x + 2} = \frac{-x - 2}{x + 2} = -1 \quad \frac{5 - x}{x - 5} = \frac{-x - 5}{x - 5} = -1
\]

Note that in all cases we assume the denominator does not equal zero.
Example 1

Simplify \( \frac{12(x-1)^3(x+2)}{3(x-1)^2(x+2)^2} \) completely.

Factor: \( \frac{4 \cdot 3(x-1)^2(x-1)(x+2)}{3(x-1)^2(x+2)(x+2)} \)

Note: \( \frac{3}{3} \cdot \frac{(x-1)^2}{(x-1)^2} \), and \( \frac{x+2}{x+2} = 1 \), so
the simplified form is \( \frac{4(x-1)}{x+2} \), \( x \neq 1 \) or -2.

Example 2

Simplify \( \frac{x^2 - 6x + 8}{x^2 + 4x - 12} \) completely.

Factor: \( \frac{x^2 - 6x + 8}{x^2 + 4x - 12} = \frac{(x - 4)(x - 2)}{(x + 6)(x - 2)} \)

Since \( \frac{x - 2}{x - 2} = 1 \), \( \frac{x - 4}{x + 6} \) is the simplified form,
\( x \neq -6 \) or 2.
Simplify each of the following rational expression completely. Assume the denominator is not equal to zero.

1. \( \frac{2(x + 3)}{4(x - 2)} \)
2. \( \frac{2(x - 3)}{6(x + 2)} \)
3. \( \frac{2(x + 3)(x - 2)}{6(x - 2)(x + 2)} \)
4. \( \frac{4(x - 3)(x - 5)}{6(x - 3)(x + 2)} \)
5. \( \frac{3(x - 3)(4 - x)}{15(x + 3)(x - 4)} \)
6. \( \frac{15(x - 1)(7 - x)}{25(x + 1)(x - 7)} \)
7. \( \frac{24(y - 4)(y - 6)}{16(y + 6)(6 - y)} \)
8. \( \frac{36(y + 4)(y - 16)}{32(y + 16)(16 - y)} \)
9. \( \frac{(x + 3)^2(x - 2)^4}{(x + 3)^4(x - 2)^5} \)
10. \( \frac{(x + 3)^4(x - 2)^5}{(x + 3)^7(x - 2)} \)
11. \( \frac{(x + 5)^4(x - 3)^2}{(x + 5)^5(x - 3)^3} \)
12. \( \frac{(2x - 5)^2(x + 3)^2}{(2x - 5)^5(x + 3)^3} \)
13. \( \frac{(5 - x)^2(x - 2)^2}{(x + 5)^4(x - 2)^3} \)
14. \( \frac{(5 - x)^4(3x - 1)^2}{(x - 5)^4(3x - 2)^3} \)
15. \( \frac{12(x - 7)(x + 2)^4}{20(x - 7)^2(x + 2)^5} \)
16. \( \frac{24(3x - 7)(x + 1)^6}{20(3x - 7)^3(x + 1)^5} \)
17. \( \frac{x^2 - 1}{(x + 1)(x - 2)} \)
18. \( \frac{x^2 - 4}{(x + 1)^2(x - 2)} \)
19. \( \frac{x^2 - 4}{x^2 + x - 6} \)
20. \( \frac{x^2 - 16}{x^3 + 9x^2 + 20x} \)

**Answers**

1. \( \frac{x + 3}{2(x - 2)} \)
2. \( \frac{x - 3}{3(x + 2)} \)
3. \( \frac{x + 3}{3(x + 2)} \)
4. \( \frac{2(x - 5)}{3(x + 2)} \)
5. \( -\frac{x - 3}{5(x + 3)} \)
6. \( -\frac{3(x - 1)}{5(x + 1)} \)
7. \( -\frac{3(y - 4)}{2(y + 6)} \)
8. \( -\frac{9(y + 4)}{8(y + 16)} \)
9. \( \frac{x - 2}{(x + 3)^2} \)
10. \( \frac{(x - 2)^4}{(x + 3)^3} \)
11. \( \frac{1}{(x + 5)(x - 3)} \)
12. \( \frac{1}{(2x - 5)^3(x + 3)} \)
13. \( \frac{(5 - x)^2}{(x + 5)^4(x - 2)} \)
14. \( \frac{(3x - 1)^2}{(3x - 2)^3} \)
15. \( \frac{3}{5(x - 7)(x + 2)} \)
16. \( \frac{6(x + 1)}{5(3x - 7)^2} \)
17. \( \frac{x - 1}{x - 2} \)
18. \( \frac{x + 2}{(x + 1)^2} \)
19. \( \frac{x + 2}{x + 3} \)
20. \( \frac{x - 4}{x(x + 5)} \)
MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS

To multiply or divide rational expressions, follow the same procedures used with numerical fractions. However, it is often necessary to factor the polynomials in order to simplify.

Example 1

Multiply \( \frac{x^2 + 6x}{(x + 6)^2} \cdot \frac{x^2 + 7x + 6}{x^2 - 1} \) and simplify the result.

After factoring, the expression becomes:

\[
\frac{x(x + 6)}{(x + 6)(x + 1)} \cdot \frac{(x + 6)(x + 1)}{(x - 1)(x + 1)}
\]

After multiplying, reorder the factors:

\[
\frac{x}{x - 1} \cdot 1 = \frac{x}{x - 1}
\]

Note: \( x \neq -6, -1, \) or 1.

Example 2

Divide \( \frac{x^2 - 4x - 5}{x^2 - 4x + 4} \div \frac{x^2 - 2x - 15}{x^2 - 4x - 12} \) and simplify the result.

First, change to a multiplication expression by inverting (flipping) the second fraction:

\[
\frac{x^2 - 4x - 5}{x^2 - 4x + 4} \cdot \frac{x^2 - 4x - 12}{x^2 - 2x - 15}
\]

After factoring, the expression is:

\[
\frac{(x - 5)(x + 1)}{(x - 2)(x - 3)} \cdot \frac{(x + 6)(x - 2)}{(x - 5)(x + 3)}
\]

Reorder the factors (if you need to):

\[
\frac{(x - 5)(x + 1)}{(x - 5)(x + 3)} \cdot \frac{(x + 6)(x - 2)}{(x - 2)(x - 3)}
\]

Since \( \frac{x - 5}{x - 5} = 1 \) and \( \frac{x + 2}{x - 2} = 1 \), simplify:

\[
\frac{(x + 1)(x + 6)}{(x + 3)} \text{ or } \frac{x^2 + 7x + 6}{x^2 + x - 6}
\]

Note: \( x \neq -3, 2, \text{ or } 5 \).

Multiply or divide each pair of rational expressions. Simplify the result. Assume the denominator is not equal to zero.

1. \( \frac{x^2 + 5x + 6}{x^2 - 4x} \cdot \frac{4x}{x + 2} \)
2. \( \frac{x^2 - 2x}{x^2 - 4x + 4} \div \frac{4x^2}{x - 2} \)
3. \( \frac{x^2 - 16}{(x - 4)^2} \cdot \frac{x^2 - 3x - 18}{x^2 - 2x - 24} \)
4. \( \frac{x^2 - x - 6}{x^2 + 3x - 10} \cdot \frac{x^2 + 2x - 15}{x^2 - 6x + 9} \)
5. \( \frac{x^2 - x - 6}{x^2 - x - 20} \cdot \frac{x^2 + 6x + 8}{x^2 - x - 6} \)
6. \( \frac{x^2 - x - 30}{x^2 + 13x + 40} \cdot \frac{x^2 + 11x + 24}{x^2 - 9x + 18} \)
7. \( \frac{15 - 5x}{x^2 - x - 6} + \frac{5x}{x^2 + 6x + 8} \)

8. \( \frac{17x + 119}{x^2 + 5x - 14} \div \frac{9x - 1}{x^2 - 3x + 7} \)

9. \( \frac{2x^2 - 5x - 3}{3x^2 - 10x + 3} \div \frac{9x^2 - 1}{4x^2 + 4x + 1} \)

10. \( \frac{x^2 - 1}{x^2 - 6x - 7} \div \frac{x^3 + x^2 - 2x}{x - 7} \)

11. \( \frac{2x - 21}{x^2 - 49} \div \frac{3x}{x^2 + 7x} \)

12. \( \frac{x^2 - y^2}{x + y} \cdot \frac{1}{x - y} \)

13. \( \frac{y^2 - y}{w^2 - y^2} \div \frac{y^2 - 2y + 1}{1 - y} \)

14. \( \frac{y^2 - y - 12}{v + 2} \div \frac{y - 4}{v^2 - 4v - 12} \)

15. \( \frac{x^2 + 7x + 10}{x + 2} \div \frac{x^2 + 2x - 15}{x + 2} \)

**Answers**

1. \( \frac{4(x + 3)}{(x - 4)} \)
2. \( \frac{1}{4x} \)
3. \( \frac{(x + 3)}{x - 4} \)
4. \( \frac{(x + 2)}{(x - 7)} \)
5. \( \frac{x + 2}{x - 5} \)
6. \( \frac{x + 3}{x - 3} \)
7. \( -\frac{x - 4}{x} \)
8. \( \frac{17(x - 1)}{9x - 1} \)
9. \( \frac{3x + 1}{2x + 1} \)
10. \( \frac{1}{x(x + 2)} \)
11. 1  
12. 1  
13. $\frac{-y}{w^2 - y^2}$  
14. $(y + 3)(y - 6)$  
15. $\frac{x + 2}{x - 3}$
Adding and Subtraction of Rational Expressions #26

Addition and subtraction of rational expressions is done the same way as addition and subtraction of numerical fractions. Change to a common denominator (if necessary), combine the numerators, and then simplify.

**Example**

The Least Common Multiple (lowest common denominator) of \((x + 3)(x + 2)\) and \((x + 2)\) is \((x + 3)(x + 2)\).

The denominator of the first fraction already is the Least Common Multiple. To get a common denominator in the second fraction, multiply the fraction by \(\frac{x + 3}{x + 3}\), a form of one (1).

Multiply the numerator and denominator of the second term:

\[
\frac{4}{(x+2)(x+3)} + \frac{2x}{x+2}
\]

Distribute in the second numerator.

\[
\frac{4}{(x+2)(x+3)} + \frac{2x(x+3)}{(x+2)(x+3)}
\]

Add, factor, and simplify. Note: \(x \neq -2\) or -3.

\[
\frac{2x^2 + 6x + 4}{(x+2)(x+3)} = \frac{2(x+1)(x+2)}{(x+2)(x+3)} = \frac{2(x+1)}{(x+3)}
\]

Simplify each of the following sums and differences. Assume the denominator does not equal zero.

1. \(\frac{7x}{3v^2} + \frac{4y}{6x^2}\)
2. \(\frac{-18}{9x} + \frac{7}{2x} - \frac{2}{3x^2}\)
3. \(\frac{7}{v^8} - \frac{6}{v-8}\)
4. \(\frac{y - 1}{v-1}\)
5. \(\frac{x}{x+3} - \frac{6x}{x^2-9}\)
6. \(\frac{6}{x^2+4x+4} + \frac{5}{x+2}\)
7. \(\frac{2a}{3a-15} + \frac{-16a+20}{3a^2-12a-15}\)
8. \(\frac{w+12}{4w-16} - \frac{w+4}{2w-8}\)
9. \(\frac{3x+1}{x^2-16} - \frac{3x+5}{x^2+8x+16}\)
10. \(\frac{7x-1}{x^2-2x-3} - \frac{6x}{x^2-x-2}\)
11. \(\frac{3}{x-1} + \frac{4}{1-x} + \frac{1}{x}\)
12. \(\frac{3y}{9y^4-4x^2} - \frac{1}{3y+2x}\)
13. \(\frac{2}{x+4} - \frac{x-4}{x^2-16}\)
14. \(\frac{5x+9}{x^2-2x-3} + \frac{6}{x^2-7x+12}\)
15. \(\frac{x+4}{x^2-3x-28} + \frac{x-5}{x^2+2x-35}\)

**Answers**

1. \(\frac{14x^3 + 4y^3}{6x^3v^2}\)
2. \(-\frac{12x + 21xy - 4y}{6x^3v}\)
3. \(\frac{13}{v-8}\)
4. \(\frac{y^2 - 2y + 2}{v-1}\)
5. \(\frac{x(x-9)}{(x+3)(x-3)}\)
6. \(\frac{5x+16}{(x+2)^2}\)
7. \(\frac{2(a-2)}{3(a+1)}\)
8. \(-\frac{1}{4}\)
9. \(\frac{4(5x+6)}{(x-4)(x+4)^2}\)
10. \(\frac{x+2}{(x-3)(x-2)}\)
11. \(\frac{-1}{x(x-1)}\)
12. \(\frac{2x}{(3v+2xY3v-2x)}\)
13. \(\frac{1}{x+4}\)
14. \(\frac{5(x+2)}{(x-4)(x+1)}\)
15. \(\frac{2x}{(x+7)(x-7)}\)

Geometry
SOLVING MIXED EQUATIONS AND INEQUALITIES

Solve these various types of equations.

1. \(2(x - 3) + 2 = -4\)
2. \(6 - 12x = 108\)
3. \(3x - 11 = 0\)

4. \(0 = 2x - 5\)
5. \(y = 2x - 3\)
   \(x + y = 15\)
6. \(ax - b = 0\)
   (solve for \(x\))

7. \(0 = (2x - 5)(x + 3)\)
8. \(2(2x - 1) = -x + 5\)
9. \(x^2 + 5^2 = 13^2\)

10. \(2x + 1 = 7x - 15\)
11. \(\frac{5 - 2x}{3} = \frac{x}{5}\)
12. \(2x - 3y + 9 = 0\)
   (solve for \(y\))

13. \(x^2 + 5x + 6 = 0\)
14. \(x^2 = y\)
   \(100 = y\)
15. \(x - y = 7\)
   \(y = 2x - 1\)

16. \(x^2 - 4x = 0\)
17. \(x^2 - 6 = -2\)
18. \(\frac{x}{2} + \frac{x}{3} = 2\)

19. \(x^2 + 7x + 9 = 3\)
20. \(y = x + 3\)
   \(x + 2y = 3\)
21. \(3x^2 + 7x + 2 = 0\)

22. \(\frac{x}{x + 1} = \frac{5}{7}\)
23. \(x^2 + 2x - 4 = 0\)
24. \(\frac{1}{x} + \frac{1}{3x} = 2\)

25. \(3x + y = 5\)
   \(x - y = 11\)
26. \(y = \frac{3}{4}x + 4\)
   \(\frac{1}{4}x - y = 8\)
27. \(3x^2 = 8x\)

28. \(|x| = 4\)
29. \(\frac{3}{2}x + 1 = \frac{1}{2}x - 3\)
30. \(x^2 - 4x = 5\)

31. \(3x + 5y = 15\)
   (solve for \(y\))
32. \((3x)^2 + x^2 = 15^2\)
33. \(y = 11\)
   \(y = 2x^2 + 3x - 9\)

34. \((x + 2)(x + 3)(x - 4) = 0\)
35. \(|x + 6| = 8\)
36. \(2(x + 3) = y + 2\)
   \(y + 2 = 8x\)

37. \(2x + 3y = 13\)
   \(x - 2y = -11\)
38. \(2x^2 = -x + 7\)
39. \(1 - \frac{5}{6x} = \frac{x}{6}\)

40. \(\frac{x - 1}{5} = \frac{3}{x + 1}\)
41. \(\sqrt{2x + 1} = 5\)
42. \(2|2x - 1| + 3 = 7\)

43. \(\sqrt{3x - 1} + 1 = 7\)
44. \((x + 3)^2 = 49\)
45. \(\frac{4x - 1}{x - 1} = x + 1\)
Solve these various types of inequalities.

46. \[4x - 2 \leq 6\]  
47. \[4 - 3(x + 2) \geq 19\]  
48. \[\frac{x}{2} > \frac{3}{7}\]

49. \[3(x + 2) \geq -9\]  
50. \[-\frac{2}{3}x < 6\]  
51. \[y < 2x - 3\]

52. \[|x| > 4\]  
53. \[x^2 - 6x + 8 \leq 0\]  
54. \[|x + 3| > 5\]

55. \[2x^2 - 4x \geq 0\]  
56. \[y \leq -\frac{2}{3}x + 2\]  
57. \[y \leq -x + 2\] \[y \leq 3x - 6\]

58. \[|2x - 1| \leq 9\]  
59. \[5 - 3(x - 1) \geq -x + 2\]  
60. \[y \leq 4x + 16\] \[y > \frac{4}{3}x - 4\]

Answers

1. 0  
2. -8.5  
3. \(\frac{11}{3}\)  
4. \(\frac{5}{2}\)

5. \((6, 9)\)  
6. \(x = \frac{b}{a}\)  
7. \(\frac{5}{2}, -3\)  
8. \(\frac{7}{5}\)

9. \(\pm 12\)  
10. \(\frac{16}{5}\)  
11. \(\frac{25}{13}\)  
12. \(y = \frac{2}{3}x + 3\)

13. -2, -3  
14. \((\pm 10, 100)\)  
15. \((-6, -13)\)  
16. 0, 4

17. \(\pm 2\)  
18. \(\frac{12}{5}\)  
19. \(-1, -6\)  
20. \(-1, 2\)

21. \(\frac{1}{3}, -2\)  
22. \(\frac{5}{2}\)  
23. \(\frac{-2 + 2\sqrt{50}}{2}\)  
24. \(\frac{2}{3}\)

25. \((4, -7)\)  
26. \((12, -5)\)  
27. \(0, \frac{8}{3}\)  
28. \(\pm 4\)

29. -24  
30. 5, -1  
31. \(y = \frac{3}{5}x + 3\)  
32. \(\approx \pm 4.74\)

33. \((-4, 11)\)  
34. \(-2, -3, 4\)  
35. 2, -14  
36. \((1, 6)\)

37. \((-1, 5)\)  
38. \(\frac{1 + \sqrt{5}}{4}\)  
39. 1, 5  
40. \(\pm 4\)

41. 12  
42. \(\frac{3}{2}, -\frac{1}{2}\)  
43. \(\frac{17}{3}\)  
44. 4, -10

45. 0, 4  
46. \(x \leq 2\)  
47. \(x \leq -7\)  
48. \(x > \frac{6}{7}\)

49. \(x \geq -5\)  
50. \(x > -9\)  
51. below  
52. \(x > 4, x < -4\)

53. \(2 \leq x \leq 4\)  
54. \(x > 2\) or \(x < -8\)  
55. \(x \leq 0\) or \(x \geq 2\)  
56. below

57. below  
58. \(-4 \leq x \leq 5\)  
59. \(x \leq 3\)  
60. below
51. $y = 3x - 6$

56. $y = -x + 2$

57. $y = 4x + 16$

60. $y = 3x - 6$